Remote Implementation of Quantum Operations

Susana F. Huelga,[1] Martin B. Plenio,[2,3] Guo-Yong Xiang,[4] Jian Li,[4] and Guang-Can Guo[4]

[1]*Quantum Physics Group, STRI,Department of Physics, Astronomy and Mathematics, University of Hertfordshire, Hatfield, Herts AL10 9AB, UK*

[2]*QOLS, Blackett Laboratory, Imperial College London, London SW7 2BW, UK*

[3]*Institute for Mathematical Sciences, Imperial College London, 53 Exhibition Road, London SW7 2BW, UK*

[4]*Key Laboratory of Quantum Information and Department of Physics, University of Science and Technology of China, Hefei 230026, China*

(Dated: July 5, 2007)

Shared entanglement allows, under certain conditions, the remote implementation of quantum operations. We revise and extend recent theoretical results on the remote control of quantum systems as well as experimental results on the remote manipulation of photonic qubits via linear optical elements.

PACS numbers: 03.67.-a, 03.67.Hk

I. INTRODUCTION

Consider a maximally entangled state of two particles A and B, for instance, the Bell state

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B). \tag{1}$$

Imagine now that the state describes a situation where the particles are in space-like separated regions and let us call Alice and Bob the remote partners with access to particles A and B respectively. We assume that Alice and Bob can perform arbitrary local operations (LO) on their subsystems, including unitary actions and, possibly, generalized measurements involving extra ancilla particles. If necessary, Alice and Bob can exchange classical communication (CC) but global quantum operations involving subsystems A and B are forbidden. The resulting set of allowed operations is generally referred to as LOCC. When Alice and Bob share entanglement, local actions on a given region have a *non-local* effect, in the sense that the state of the remote particle does not necessarily remain insensitive to the details of the transformation performed miles away. Consider the case of a projective measurement on Alice's side. For example, she performs a Stern-Gerlach measurement with the apparatus oriented along an arbitrary unit vector $\hat{n}$. Let us call $|+\rangle_{\hat{n}}$, the eigenvector corresponding to spin up along the $\hat{n}$ direction. Whenever she records a spin up result, the joint state of the system is projected out into the product state $|+\rangle_{\hat{n},A}|+\rangle_{\hat{n},B}$, if she records a spin down result, the joint state of the system is projected out into the product state $|-\rangle_{\hat{n},A}|-\rangle_{\hat{n},B}$. Both outcomes happen with equal probability. Now there are two distinctly different ways to proceed. If Bob were to conduct an independent

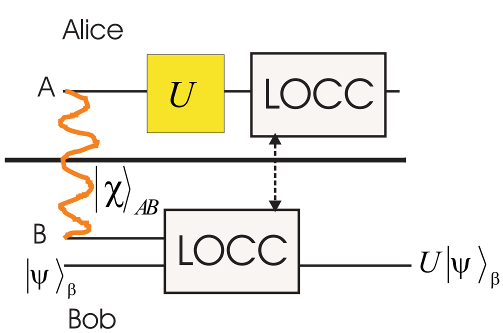

FIG. 1: Two remote partners share bipartite entanglement, represented by the state $|\chi\rangle_{AB}$. Only one of the partners, Alice, has access to a device that can implement an arbitrary single qubit operation U. Is there an LOCC protocol that can yield a final state where Bob ends up holding the state $U|\psi\rangle$ for any $|\psi\rangle$?

measurement on his particles using an Stern-Gerlach apparatus with a random orientation, then he will also register a random outcome, obtaining an unbiased spin up and spin down distribution along his chosen direction. There will be no correlations between his measurement outcomes and those of Alice. If however, Bob receives a message from Alice informing him of her choice of alignment, then the situation changes in a subtle way. His measurement will still give random outcomes, but, these outcomes will be perfectly correlated to those of Alice's measurement outcomes. This shows that classical communication may be used to correlate actions between partners with the effect that experimental results will change significantly. In a more complicated setting Alice and Bob would perform measurements along various non-parallel but correlated axes. This type of correlated measurements then leads to the celebrated EPR paradox, later formulated by Bell in a testable form and which for years fuelled the discussions about quantum non-locality [1]. The recent development of the theory of quantum information processing has given an unexpected twist to this discussion. Shared entanglement supplement by LOCC operations makes possible entirely new forms of distributed computation and communication [2], as epitomized by the process of quantum state teleportation [3]. In this article we will focus on a related problem: The remote implementation of quantum operations, and in particular, the teleportation of unitary transformations [4, 5]. The simplest situation has been illustrated pictorially in Figure 1. One of the remote partners, Alice, has access to a classical device that can implement an arbitrary singe-particle operation U. There is no need to restrict the dimensionality of the space of allowed operations except for assuming that it be finite, but possibly very large (See section I for specific details). Using LOCC operations, and provided that Alice and Bob share entanglement, represented by the state $|\chi\rangle_{\alpha AB}$ in the figure, our aim is to remotely implement the arbitrary transformation U on Bob's side, i.e., to design an LOCC protocol that will end up with Bob holding the state $U|\psi\rangle_\beta$ for any single particle state $|\psi\rangle_\beta$. When we allow for a completely arbitrary U then we will show that the most economical procedure to achieve this consist of teleporting the state of Bob's particle to Alice, who applies the transformation U to the teleported state and then teleports the state back to Bob. No other LOCC protocol supplemented by shared entanglement will consume less distributed entanglement and have a lower communication cost. However, when the requirement of perfect remote control for arbitrary operations is relaxed, we encounter a different situation and state teleportation is no longer the most cost-efficient way to proceed when implementing restricted sets of operations. Non-trivial protocols become possible and we present examples for these in the special case of unitary transformations on two-level systems (qubits).

We have organized the presentation as follows. In Section II we present a general proof for the necessary and sufficient resources required for the remote control of a single qudit operation. This results generalize previous constraints derived for qubits [4]. Section III analyzes the limitations arising when one attempts to teleport restricted sets of operations. We will show that arbitrary qubit rotations around a fixed direction can be implemented remotely without the need of teleporting states between Alice and Bob. Novel results concerning the remote control of identical copies are discussed in III.a. Recent experimental reports demonstrating the protocol are summarized in section VI. The final section summarizes our main results and conclusions.

II. A GENERAL RESULT

The most general scenario for the teleportation of an arbitrary unitary operation was discussed in [4] and is related to the design of universal quantum gates arrays proposed by Nielsen and Chuang [6]. When the device implementing the unitary transformation U is modelled as a truly quantum system, whose state corresponding to the unitary operation U will be denoted by $|U\rangle_C$, we can represent the remote control operator by a completely positive, linear, trace preserving map on the set of density operators for the combined system. Any such map has a unitary representation $\mathcal{G}$ involving global ancillary systems in state $|\chi\rangle_{AB}$, so that

$$\mathcal{G}\left[|\chi\rangle_{AB}\otimes|U\rangle_C\otimes|\psi\rangle_\beta\right] = |\Phi(U,\chi)\rangle_{ABC}\otimes\left(U|\psi\rangle_\beta\right) \;.. \tag{2}$$

Note that this unitary representation may be non-local even if the map itself is local. As a consequence, any argument involving the eq. (2) will provide only lower bounds on the resource requirement. In any specific case one will then need to find a local protocol that matches these lower bounds. This can indeed be done in any of the instances that we are considering in the following. The results by Nielsen and Chuang on programmable gate arrays [6] imply that the control states $|U\rangle_C$ corresponding to different unitary transformations are orthogonal, so that no finite-dimensional control system can be used to teleport an arbitrary unitary operation. For the remainder of this paper, when we speak of an arbitrary unitary operation, we will mean one which belongs to some arbitrarily large, but finite, set. We will also assume that this set contains the identity $\sigma^0 = \mathbb{1}$ and the 3 Pauli operators σ^i. The orthogonality of the control states opens the possibility that different operations can, at least in principle, be distinguished and identified by Alice if she chooses to perform measurements on the apparatus. Here we are not interested in this approach and focused instead in assisting the task with shared entangled and LOCC operations [7].

Within this very general formulation, it is possible to derive lower bounds on the amount of non-local resources that are needed to implement $\mathcal{G}$ using only local operations and classical communication. To this end, we should remember the following basic principles [4, 8]

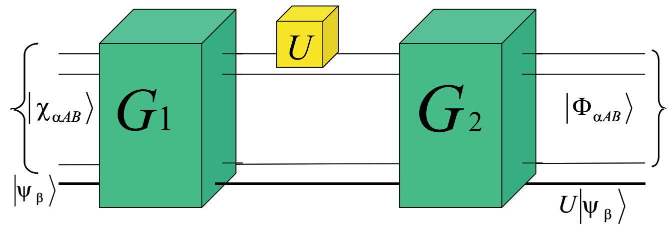

FIG. 2: Quantum circuit model using nonlocal unitary gates G_1 and G_2 for the teleportation of a unitary operation U on a single particle (See text for details). Alice and Bob have access to the two upper and the two lower wires respectively. The lowest wire represents the selected single particle state onto which we intend to remotely apply the transformation U.

(a) *The amount of classical information able to be communicated by an operation in a given direction across some partition between subsystems cannot exceed the amount of information that must be sent in this direction across the same partition to complete the operation* (Impossibility of super-luminal communication).

(b) *The amount of bipartite entanglement that an operation can establish across some partition between subsystems cannot exceed the amount of prior entanglement across the partition that must be consumed in order to complete the operation*(Impossibility of increasing entanglement under LOCC).

Principle (a) allows us to establish the fact that at least two classical bits must be sent from Alice to Bob to complete the teleportation of an arbitrary U. Moreover, by teleporting an arbitrary U according to the general prescription in Eq. (2), Alice and Bob can establish 2 ebits of shared entanglement and it follows from principle (b) that at least 2 ebits of entanglement need to be consumed to implement $\mathcal{G}$ locally, i.e. to teleport an arbitrary unitary operation [4, 5]. These bounds can be attained by a procedure in which Bob teleports the state of his particle to Alice who, after applying the unitary transformation, teleports it back to him (bidirectional state teleportation). This scheme saturates the lower bounds for the amount of shared ebits and classical bits transmitted from Alice to Bob and additionally uses two bits of classical communication from Bob to Alice and allows the faithful implementation of U independently of the dimension of the control system. To be more efficient overall, any other scheme would need less resources than bidirectional state teleportation. This establishes an upper bound in the overall amount of resources required for the efficient remote implementation of an arbitrary U as 4 classical bits and 2 ebits.

Note that it would also be conceivable to adopt a different strategy – teleporting the state of the control system from Alice to Bob who would then implement the control directly onto qubit β. We call this the "control-state teleportation" scheme. Control-state teleportation is a unidirectional communication scheme from Alice to Bob, so the absolute lower bound for the communication exchange from Bob to Alice is zero. Obviously, the overall resources will depend on the dimensionality of the control system C and in general a large amount of entanglement and classical communication from Alice to Bob will be required if we want to teleport the control system.

Let us focus on the experimental scenario where the black box implementing an arbitrary transformation U is a macroscopic object, involving a (very) large number of degrees of freedom. The option of teleporting the control apparatus is then unfeasible, given that it would consume an unrealistic amount of entanglement and classical communication resources. However, the question remains whether there exists a more economical protocol than bidirectional state teleportation. We will generalize in the following the results presented in [4] and show that bidirectional state teleportation is an unconditional optimal way to remotely implement an arbitrary U on a given d-level system (qudit).

Discarding the possibility of control-state teleportation allows us to replace the transformation given by Eq. (2) with

$$G_2\, U\, G_1(|\chi\rangle_{\alpha AB} \otimes |\psi\rangle_\beta) = |\Phi(U,\chi)\rangle_{\alpha AB} \otimes U|\psi\rangle_\beta, \tag{3}$$

where certain fixed operations G_1 and G_2 are performed, respectively, prior to and following the action of the arbitrary U on a qubit α on Alice's side, as illustrated in Figure 2. We assume that Alice and Bob share initially some entanglement, represented by the state $|\chi\rangle_{\alpha AB}$. As before, the purpose of the transformation is to perform the operation U on Bob's qubit β. Note that we chose to use a nonlocal unitary representation of the transformation involved so that G_1 and G_2 are unitary operators acting on possibly all subsystems. For instance, the transformation G_i can represent a state teleportation operation. In the following we prove, for systems of arbitrary spatial dimensions, that this is necessarily the case and that the only way that Eq. (3) can be implemented (locally) is by teleporting the state $|\psi\rangle_\beta$ from Bob to Alice, and then teleporting back the transformed state $U|\psi\rangle_\beta$ from Alice to Bob.

By linearity, the transformed state of systems αAB has to be independent of the particular input state $|\psi\rangle_\beta$. and the specific operation U [4]. With this we can already show that the operation G_1 cannot be trivial. We do this by first assuming the contrary that $G_1 = \mathbb{1}$, and considering two input states, $|\psi\rangle_\beta$ and $|\psi'\rangle_\beta$ such that ${}_\beta\langle\psi^{'}|\psi\rangle_\beta = 0$, and two unitary transformations U and U' which bring these two states to the same state $|\gamma\rangle_\beta$. Using Eq. (3), this implies that

$$\begin{aligned} G_2\left(U|\chi\rangle_{\alpha AB}\,|\psi\rangle_\beta\right) &= |\Phi(\chi)\rangle_{\alpha AB}\otimes|\gamma\rangle_\beta \\ G_2\left(U'|\chi\rangle_{\alpha AB}\,|\psi'\rangle_\beta\right) &= |\Phi(\chi)\rangle_{\alpha AB}\otimes|\gamma\rangle_\beta\ .. \end{aligned} \tag{4}$$

No universal unitary action G_2 can be found to satisfy Eq. (4), as this would require the mapping of orthogonal states onto the same state. This shows that no universal operation G_2 that satisfies Eq. (4) can exist and therefore, for the U-teleportation to succeed, G_1 has to be non-trivial. Finally, we will show that each of the operations G_i implements a state transfer. Let us rewrite Eq. (3) as

$$U\,G_1(|\chi\rangle_{\alpha AB}\otimes|\psi\rangle_\beta) = G_2^\dagger(|\Phi(\chi)\rangle_{\alpha AB}\otimes U|\psi\rangle_\beta). \tag{5}$$

Let us denote by $\{|k\rangle\}_{k=0}^{d-1}$ the canonical basis in the space of sates of qudits. Consider the hermitean operator Π defined as

$$\Pi = \sum_{k=0}^{d-1}(k+1)|k\rangle\langle k|. \tag{6}$$

By construction Π is diagonal in the canonical basis and has a non-degenerate spectrum. Since the operations G_1 and G_2 are universal, we may choose U and $|\psi\rangle_\beta$ freely. For each $|\psi\rangle_\beta$ let the operator U_ψ be such that $U_\psi|\psi\rangle = |0\rangle$ where $\Pi|0\rangle = |0\rangle$. If $U = \Pi U_\psi$, then

$$\begin{aligned} (\Pi U_\psi)\,G_1\Big(|\chi\rangle_{\alpha AB}\otimes|\psi\rangle_\beta\Big) &= G_2^\dagger\Big(|\Phi(\chi)\rangle_{\alpha AB}\otimes\Pi U_\psi|\psi\rangle_\beta\Big) \\ &= G_2^\dagger\Big(|\Phi(\chi)\rangle_{\alpha AB}\otimes|0\rangle_\beta\Big)\ . \end{aligned}$$

The RHS is simply $(U_\psi)G_1\left(|\chi\rangle_{\alpha AB}\otimes|\psi\rangle_\beta\right)$ and so, given that the spectrum of the operator Π is not degenerate, $(U_\psi)G_1\left(|\chi\rangle_{\alpha AB}\otimes|\psi\rangle_\beta\right)$ is the eigenstate $|0\rangle_\alpha\otimes|\phi\rangle_{AB\beta}$ of $(\Pi)_\alpha\otimes 1_{AB\beta}$. Equivalently,

$$\begin{aligned} G_1\left(|\chi\rangle_{\alpha AB}\otimes|\psi\rangle_\beta\right) &= \left(U_\psi^\dagger|0\rangle_\alpha\right)\otimes|\phi\rangle_{AB\beta} \\ &= |\psi\rangle_\alpha\otimes|\phi\rangle_{AB\beta}\ . \end{aligned} \tag{7}$$

In other words, the operation G_1 necessarily transfers Bob's state $|\psi\rangle$ to Alice's qubit α. Substituting Eq. (7) into Eq. (3) then shows that G_2 necessarily transfers $U|\psi\rangle$ back to Bob's qubit β. This implies that the state of Bob's qubit must be brought to Alice for it to be acted on by the local operator U. This results is tantamount to a no-go theorem: *a local unitary operation U cannot act remotely.* From this and the fact that quantum state teleportation is an optimal procedure for local state transfer, we conclude that the optimal LOCC procedure supplemented by shared entanglement for implementing remotely an arbitrary unitary action U on a qudit is by means of bidirectional state teleportation.

III. RESTRICTED OPERATIONS ON QUBITS: TELEPORTING ARBITRARY ROTATIONS AROUND A FIXED AXIS

In the specific case of qubits, the results presented in the previous section show that if we want the transformation U to be an arbitrary element of the group $SU(2)$, no LOCC protocol can exist consuming less overall resources than teleporting Bob's state to Alice followed by Alice teleporting the state transformed by U back to Bob. This amounts to two e-bits of entanglement and two classical bit in each direction. The ultimate reason for this result can be found in the linearity of quantum mechanics and the impossibility of implementing remotely an arbitrary U without resorting to state transfer belongs to the family of no-go results imposed by the linear structure of quantum mechanics such as the non-cloning theorem [9].

The same way that the no-cloning theorem forbids the replication of general states, should we expect a similar result if the requirement of being able to implement *any* U is relaxed? Can we find families of operators that can be implemented consuming less overall resources than a two-way teleportation protocol?. We want the procedure to work with perfect efficiency. Imperfect storage of quantum operations have been recently discussed by Vidal et al. [10]. The probabilistic implementation of universal quantum processors was discussed by Nielsen and Chuang [6] and more recently by Hillery et al in the general case of qudits [11]. We will show that there are indeed two restricted classes of operations that can be implemented remotely and deterministically using less overall resources than bidirectional quantum state teleportation and only two (up to a local change of basis). These are arbitrary rotations around a fixed direction $\vec{n}$ and rotations by a fixed angle around an arbitrary direction lying in a plane orthogonal to $\vec{n}$ [5]. It is easy to show that if a given protocol would be able to implement remotely certain operation U, the same protocol would also allow the implementation of a remote control-U. This argument will allow us to establish a lower bound on the amount of classical communication that needs to be conveyed from Bob to Alice. Consider the case of a non-local controlled-NOT gate between Alice and Bob [8, 12]. When Alice prepares the state $|+\rangle_c = (|0\rangle + |1\rangle)/\sqrt{2}$, the action of a controlled-NOT gate with Bob qubit being in either state $|+\rangle_B$ or in state $|-\rangle_B$ is given by

$$\begin{aligned} |+\rangle_c|+\rangle_B &\longmapsto |+\rangle_c|+\rangle_B, \\ |+\rangle_c|-\rangle_B &\longmapsto |-\rangle_c|-\rangle_B. \end{aligned} \tag{8}$$

Therefore, this operation allows Bob to transmit one bit of information to Alice and, as a consequence, the teleportation of U requires at least one bit of communication from Bob to Alice. Summarizing, the physical principles of non-increase of entanglement under LOCC and the impossibility of super-luminal communication allow us to establish lower bounds in the resources required for teleporting an unknown quantum operation on a qubit. At least two e-bits of entanglement have to be consumed and, in addition, this quantum channel has to be supplemented by a *two way* classical communication channel which, in principle, could be non-symmetric. While consistency with causality requires two classical bits being transmitted from Alice to Bob, the lower bound for the amount of classical information transmitted from Bob to Alice has been found to be one bit.

An explicit protocol saturating these bounds was presented in [5]. This protocol allows the remote implementation of arbitrary rotations around a given axis $\hat{n}$ as well as fixed rotations around an arbitrary direction within a plane orthogonal to $\hat{n}$. Choosing $\hat{n}$ to be along the z-direction, operations of the form

$$U_{com} = \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix} = e^{i\frac{\phi}{2}\sigma_z}, \tag{9}$$

with $a = e^{i\phi}$ that is, the set of operations that commute with σ_z, or transformations of the form

$$U_{anticom} = \begin{pmatrix} 0 & b \\ -b^* & 0 \end{pmatrix} = \sigma_x e^{i(\frac{\phi+\pi}{2})\sigma_z}, \tag{10}$$

which anticommute with σ_z, i.e., are linear combinations of the Pauli operators σ_x and σ_y. Any operation within this family can be teleported deterministically using a protocol which employs less resources than bidirectional state teleportation. Remarkably, these are the only sets that have this property as we rigourously showed in [5]. The entanglement and communication costs can be further reduced if it is a priory known whether the operation U to be teleported belongs to either the set characterized by eq. (9) or to the set characterized by eq. (10). Figure 3 depicts the quantum circuit representation of the protocol that allows the deterministic remote implementation of arbitrary rotations around the z-axis. This is a communication symmetrical protocol where one classical bit is conveyed in each direction and with an overall entanglement cost of 1 ebit. As always, wiggly lines represent shared entanglement across the space-like thick solid line. The dotted arrow lines represent the exchange of classical communication following the measurement of the observable enclosed in the half-ovoid symbol. Squared boxes denote unitary actions performed after that exchange.

Related results have been recently derived by other authors [13]. Reznik et al studied the deterministic remote implementation of a class of operations whose complete specification is split between the remote partners. For instance, single qubit transformations of the form $U = e^{i\alpha_A \sigma_{\hat{n}_B}}$, where the rotation angle α_A is controlled by Alice while Bob selects the direction $\hat{n}_B$. In this case, the most economical protocol for the remote implementation of U consumes 1 ebit and requires the symmetrical exchange of two classical bits. Reznik and Groisman have also analyzed the probabilistic remote implementation of controlled operations using partially entangled states as a resource [14]. The potential use of this type of protocols for secret sharing schemes was discussed by Gea-Banacloche and C-P Yang [15].

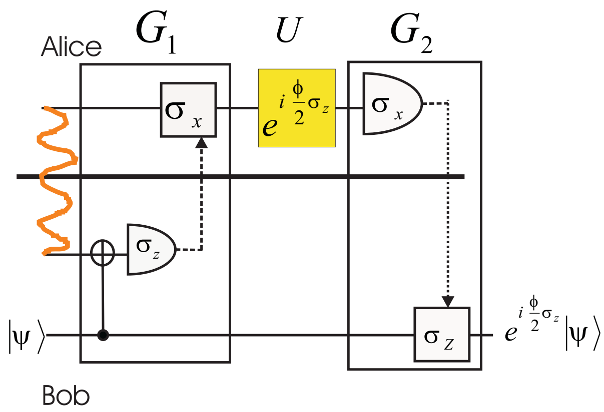

FIG. 3: Quantum circuit for the remote rotation of a single qubit. The whole process is divided into three steps whose experimental implementation using photonic qubits will be described in section IV. The operation G_1 makes the coefficients of Bob's state visible in the channel via a global action on Bob's side and subsequent measurement in the computational basis. One classical bit has to be conveyed from Alice to Bob. For instance, if Alice and Bob share initially the Bell state $|\phi\rangle^+_{AB}$ and $|\psi\rangle = \alpha|0\rangle_B + \beta|1\rangle_B$, after completing G_1 they share the state $\alpha|00\rangle_{AB} + \beta|11\rangle_B$. Next, Alice implements the rotation U on her side. Operation G_2 involves a measurement in the rotated basis followed by the transmission of a classical bit to Bob, who completes the protocol by applying a final σ_z operation conditional on Alice's measurement result.

A. Efficient application of multiple instances of an unknown unitary operation

In this subsection we are going to present a novel result concerning the generalization of quantum remote control. So far we have considered the question of a single remote application of an unknown unitary operation on an individual quantum state. One might consider whether the remote application of U on two identical copies of a quantum state can be carried out with fewer resources than two full ebits and two classical bits of communication in each direction (ie twice the resource of a single application). In the following we will show that indeed, as long as the two copies of the quantum state $|\psi\rangle$ are both held by Bob a resource reduction can be achieved. In other words, assuming Alice holds a machine that implements $U = e^{i\theta\sigma_z}$ with an unknown θ and given that Bob holds the state $|\psi\rangle^{\otimes 2}$ we would like to provide an entangled state shared between Alice and Bob which, when supplemented by local operations and classical communication, allows for Bob to hold, at the end of the protocol, the state $(U|\psi\rangle)^{\otimes 2}$. In the following we will provide a protocol that requires an entangled state with $\log 3$ ebits of entanglement. This is clearly less than 2 ebits and therefore represents a resource reduction over the trivial protocol.

The entangled resource is the well-known tele-cloning state [16]

$$|\psi_{tele}\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle_A|00\rangle_B + \frac{|01\rangle_A + |10\rangle_A}{\sqrt{2}}\frac{|01\rangle_B + |10\rangle_B}{\sqrt{2}} + |11\rangle_A|11\rangle_B\right) \tag{11}$$

For the following it will be convenient to use the following abbreviations

$$\begin{aligned} |\tilde{0}\rangle &= |00\rangle \\ |\tilde{1}\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\ |\tilde{2}\rangle &= |11\rangle \end{aligned}$$

Then the total state including $|\psi\rangle^{\otimes 2}$ with $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is given by

$$|\psi_{tele}\rangle|\psi\rangle^{\otimes 2} = \frac{1}{\sqrt{3}}(|\tilde{0}\rangle_A|\tilde{0}\rangle_B + |\tilde{1}\rangle_A|\tilde{1}\rangle_B + |\tilde{2}\rangle_A|\tilde{2}\rangle_B)(\alpha^2|\tilde{0}\rangle + \sqrt{2}\alpha\beta|\tilde{1}\rangle + \beta^2|\tilde{2}\rangle) \tag{12}$$

let us further define the three unitary transformations

$$1 = |\tilde{0}\rangle\langle\tilde{0}| + |\tilde{1}\rangle\langle\tilde{1}| + |\tilde{2}\rangle\langle\tilde{2}| \tag{13}$$
$$T = |\tilde{0}\rangle\langle\tilde{1}| + |\tilde{1}\rangle\langle\tilde{2}| + |\tilde{2}\rangle\langle\tilde{0}| \tag{14}$$
$$T^2 = |\tilde{0}\rangle\langle\tilde{2}| + |\tilde{1}\rangle\langle\tilde{0}| + |\tilde{2}\rangle\langle\tilde{1}| \tag{15}$$

and the corresponding controlled operation

$$U_T = \sum_{k=0}^{2} |\tilde{k}\rangle\langle\tilde{k}| \otimes T^k \tag{16}$$

We furthermore define the unitary operator

$$\mathcal{F} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2i\pi/3} & e^{4i\pi/3} \\ 1 & e^{4i\pi/3} & e^{8i\pi/3} \end{pmatrix} \tag{17}$$

It is now straightforward to verify that the following protocol will implement U on the two copies of $|\psi\rangle$. (i) Apply U_T between the first (second) copy of $|\psi\rangle$ as control and the first (second) qubit of Bob's part of the telecloning state as target. (ii) Measure Bob's qubits that are part of the telecloning state. If he finds $|\tilde{k}\rangle$, then apply operation T^k on Alice's qubits. (iii) Now she applies U to both of her qubits. (iv) Alice applies $\mathcal{F}$ and subsequently performs a measurement in the $\{|\tilde{0}\rangle, |\tilde{1}\rangle, |\tilde{2}\rangle\}$ basis. (v) If Alice finds $|\tilde{k}\rangle$, then Bob needs to apply the operation $\mathcal{F}^k$ on his qubits. The outcome of this procedure is the state $(U|\psi\rangle)^{\otimes 2}$ on Bob's side.

The procedure requires the transmission of $\log 3$ classical bits in both directions and furthermore requires the telecloning state as a resource which contains $\log 3$ ebit of entanglement.

Note however, that in this protocol is is strictly necessary that Bob holds both copies of the state $|\psi\rangle$ and possesses the ability to carry out joint operations on both of them. These joint operations would require require further entangled resources if the two particles on Bob's side are distant. It is therefore natural to consider the question of whether one can remotely apply the unitary operation U that Alice possesses onto two identical copies of the state $|\psi\rangle$ that are held by Bob and Charles. If one admits a 50% success rate then this is indeed possible employing a straightforward generalization of the protocol for remote application of $U = e^{i\theta\sigma_z}$ presented earlier in this section. It is an interesting open question whether there is a protocol that achieves 100% success probability *and* requires less than a shared ebit between Alice and Bob and another shared ebit between Alice and Charles.

IV. RECENT EXPERIMENTAL RESULTS

Recently, the quantum remote control protocol for arbitrary single qubit rotations, shown in Figure 3, was implemented experimentally in a linear optics setup [17]. For that, polarization entangled states generated from spontaneous parametric down conversion (SPDC) where locally manipulated to generate a three qubit state involving polarization and path degrees of freedom. In the following we will describe the basic elements of this experiment as well as the results that have been obtained with this set-up.

We begin with a brief description of the experimental choice of the physical qubit, the gates implementing the local operations and the entangled states employed.

- *Qubits:* For photons, both horizontal and vertical polarization states $\{|H\rangle, |V\rangle\}$ as well as up and down paths $\{|u\rangle, |d\rangle\}$ can represent the logic states $\{|0\rangle, |1\rangle\}$ of qubits. We will refer to the resulting encoding as polarization or path qubits respectively.

- *Quantum gates:* For a polarization qubit, arbitrary unitary rotation can be performed by using half-wave plate(HWP) and quarter-wave plate(QWP)[18]. The controlled-NOT gate between polarization qubit (control) and path qubit (target) of the same photon can be implemented by a polarization beam splitter(PBS),

$$\begin{aligned} |H\rangle |u\rangle (|H\rangle |d\rangle) &\rightarrow |H\rangle |u'\rangle (|H\rangle |d'\rangle), \\ |V\rangle |u\rangle (|V\rangle |d\rangle) &\rightarrow |V\rangle |d'\rangle (|V\rangle |u'\rangle), \end{aligned} \tag{18}$$

  with suitable definition of the incoming and outgoing modes.

- *Entangled states:* Bi-photon entangled states involving either polarization or path qubits can be generated via a SPDC process [19]. Here the initial three qubit state of the protocol represented in Fig. 3 will involves the polarization degree of freedom of Alice's qubit and the polarization and path degrees of freedom of Bob's particle, as explained in detail below.

The concrete experimental setup that was implemented in [17] is showed schematically in Fig. 4. A mode-locked Ti:Sapphire pulsed laser (with the pulse width less than 200 fs, a repetition rate of about 82MHz and a central

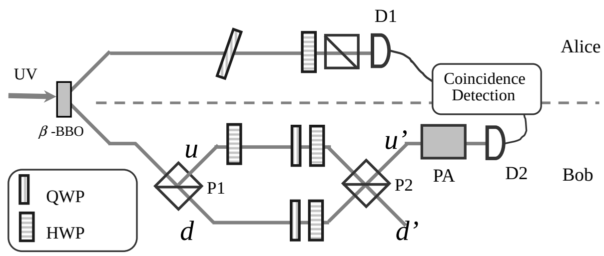

FIG. 4: Schematic representation of the experimental setup to perform a remote rotation on a single photon [17]. *P1*, *P2* denote polarization beam splitters; HWP are half wave plates; QWP are quarter wave plates; PA is a polarization analyzer; D_1, D_2 represent single photon detectors.

wavelength of 780.0nm) is frequency-doubled to produce the pumping source for a SPDC process. A BBO crystal of 1mm thickness, cut for type-II phase matching, is used as a down converter. Non-collinear degenerated SPDC generates two photons, A and B, in the polarization-entangled state

$$\left|\Psi^+\right\rangle_{AB} = \frac{1}{\sqrt{2}}(\left|H\right\rangle_A \left|V\right\rangle_B + \left|V\right\rangle_A \left|H\right\rangle_B)$$

[19]. Bob employs the PBS (denoted *P1*) to split photon B in two paths $\{\left|u\right\rangle, \left|d\right\rangle\}$ and a HWP *H1* at 45° as a σ_x gate is used to flip the polarization in path u. Hence the initial polarization entanglement between the two distributed photons is converted into polarization-path entanglement,

$$\left|\Psi^+\right\rangle_{123} = \frac{1}{\sqrt{2}}(\left|H\right\rangle_1 \left|u\right\rangle_2 + \left|V\right\rangle_1 \left|d\right\rangle_2) \left|H\right\rangle_3, \tag{19}$$

where we have relabelled Alice's photon with the index 1 and the the indices 2 and 3 refer to the path and polarization degree of freedom of photon B. The polarization state of qubit 3 can be prepared in an arbitrary state $\left|\psi\right\rangle_3 = \alpha\left|H\right\rangle_3 + \beta\left|V\right\rangle_3$ with identical sets of waveplates, $\{H_u, Q_u\}$ and $\{H_d, Q_d\}$, in each path [18]. The global state can therefore be initialized to be of the general form

$$\left|\Phi^+\right\rangle_{12} \left|\psi\right\rangle_3 = \frac{1}{\sqrt{2}}(\left|H\right\rangle_1 \left|u\right\rangle_2 + \left|V\right\rangle_1 \left|d\right\rangle_2)(\alpha\left|H\right\rangle_3 + \beta\left|V\right\rangle_3). \tag{20}$$

The three step protocol for the remote rotation of Bob's polarization qubit is performed as follows:

i) Encoding (Operation G_1): The paths u and d of photon B provide the input for a second PBS (denoted by *P2*) to perform a *CNOT* operation, where the polarization acts as the control qubit and the path represents the target qubit. In the experiment we have ensured that the optical path lengths of u and p are equal to avoid the accumulation of a relative phase factor between the two terms in eq. (19). The σ_z measurement on qubit *2* is implemented by reading out the path information of photon *B*. If B is in path u', $|\psi\rangle_B$ is encoded into $|\psi\rangle_{AB} = \alpha\,|H\rangle_A\,|H\rangle_B + \beta\,|V\rangle_A\,|V\rangle_B$. If B is in path d', the two photons will be in $|\psi'\rangle_{13} = \alpha\,|V\rangle_1\,|H\rangle_3 + \beta\,|H\rangle_1\,|V\rangle_3$, which can be transformed into $|\Psi\rangle_{13}$ by another HWP at 45° acting on photon A. Here we omit the later case without loss of generality.

ii) Remote operation: The operation U_{com} can be performed by a pair of QWP at 45° with a HWP at $\frac{\varphi}{2} - 45°$ between them. Such device has been used to verify the geometric phase of classical light and photons[20, 21]. For a single qubit operation, any additional global phase is trivial, so U_{com} can be replaced by $e^{i\varphi/2}U_{com}$, which can be realised by one zero-order waveplate at 0° tilted in a suitable angle (see [22] for similar application). Here we chose $\varphi = 60°$ and 120° by a tilted QWP *Q1*.

iii) Decoding and Verification (Operation G_2): Alice performs her measurement in the rotated basis $\{|D\rangle_1 = \frac{1}{\sqrt{2}}(|H\rangle_1 + |V\rangle_1), |C\rangle_1 = \frac{1}{\sqrt{2}}(|H\rangle_1 - |V\rangle_1)\}$ using a polarizer. Photon A is detected by a single photon detector (SPCM-AQR-14 by EG&G). Photon B will be collapsed into $|\psi'\rangle_3 = U_{com}\,|\psi\rangle_3$ for result $|+\rangle_1$, and $|\psi''\rangle_3 = U_{com}\sigma_z\,|\psi\rangle_3$ for result $|-\rangle_1$. The latter can be converted into $|\psi'\rangle_B$ by a HWP at 0°, i.e. a σ_z rotation. The polarization state of photon B is reconstructed by quantum state tomography using a polarization analyzer and a detector [18]. The measurements on A and B are collected via coincidence counts with a window time of 5ns.

Results:

The effect of a general quantum operation on a single qubit can represented by a trace-preserving completely positive (CP) map, i.e. for an arbitrary input state ρ, the output one would be of the form $\rho' = \varepsilon(\rho) = \sum_{mn}\chi_{mn}E_m\rho E_n^\dagger$, $(\{E_m\} = \{I, \sigma_x, \sigma_y, \sigma_z\})$, where χ is a positive Hermitian matrix. In the actual experiment, two main sources of phase decoherence were identified (for a more detailed discussion see [17]). One is caused by the bi-refringency of BBO, which induces the partial time-separation between the wave-packets of two polarizations. The other one is the mismatch of spatial modes in the PBS *P2*. The CP map representing the dephasing operation can be written as ε_d=$\{\sqrt{\frac{1+p\eta}{2}}U_{com}, \sqrt{\frac{1-p\eta}{2}}U_{com}\sigma_z\}$ where p is the visibility of the entangled state obtained from SPDC and η is the visibility of the interferometer formed by the PBSs $P1$ and $P2$. These parameters can be measured independently. Within this formalism, the final state after the action of the net quantum operation, including dephasing, representing the remote rotation protocol is given by

$$\rho_d = \varepsilon_d(|\psi\rangle\langle\psi|) = \begin{pmatrix} \alpha\alpha^* & p\eta\alpha\beta^*e^{-i\varphi} \\ p\eta\alpha^*\beta e^{i\varphi} & \beta\beta^* \end{pmatrix}. \tag{21}$$

Here, to completely characterize the remote operation ε_e in our experiment, four states $\{|H\rangle, |V\rangle, |D\rangle, |R\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i\,|V\rangle)\}$ are used as an input for the initial state of Bob's polarization qubit. With the four output density matrices, the χ obtained in the process can then be reconstructed using techniques for quantum process tomography [23]. For the σ_z-rotation $U_{com} = \cos\phi/2 + i\sin\phi/2\sigma_z$, χ_i has four non-zero elements, $\chi_{11} = (1+\cos\phi)/2$, $\chi_{44} = (1-\cos\phi)/2$, $\chi_{14}^* = \chi_{41} = i\sin\phi/2$. The dephasing only changes the value of the four non-zero elements, $\chi_{11} = (1 + p\,\eta\,cos\phi)/2$, $\chi_{44} = (1 - p\,\eta\,cos\phi)/2$, $\chi_{14}^* = \chi_{41} = ip\,\eta\,sin\phi$, while leaving the other 12 zero elements unaffected. The matrices are shown in Fig. 5 for the case of an ideal rotation χ_i, a dephased rotation χ_d, and the effective operation χ_e obtained in our experiment, where the left six histograms are the real parts for the matrices for 60° and 120° rotation through z axis, and the right for the imaginary ones, where the values of the parameter ε_d was measured to be $p \approx 0.85$ and $\eta \approx 0.92$. New non-zero elements are found in the effective χ_e. This is introduced by the imperfection of the polarization beam-splitter. The comparison of the experimental operation ε_e with the ideal rotation ε_i is determined by evaluating the average fidelity with pure input states uniformly distributed over the Bloch sphere

$$\overline{F}[\varepsilon_e, U_{com}] = \int d\psi F[\varepsilon_e(|\psi\rangle, U_{com}|\psi\rangle\langle\psi|U_{com}^\dagger]$$

where

$$F[\rho, \rho'] = (Tr[\sqrt{\sqrt{\rho'}\rho\sqrt{\rho'}}])^2$$

is the output state fidelity [24, 25]. The measured χ yields $\overline{F}_{60} = 0.96$ and $\overline{F}_{120} = 0.86$. Although only a rotation commuting with σ_z (z-rotation) is operated in our experiment, the same protocol can be implemented for operations

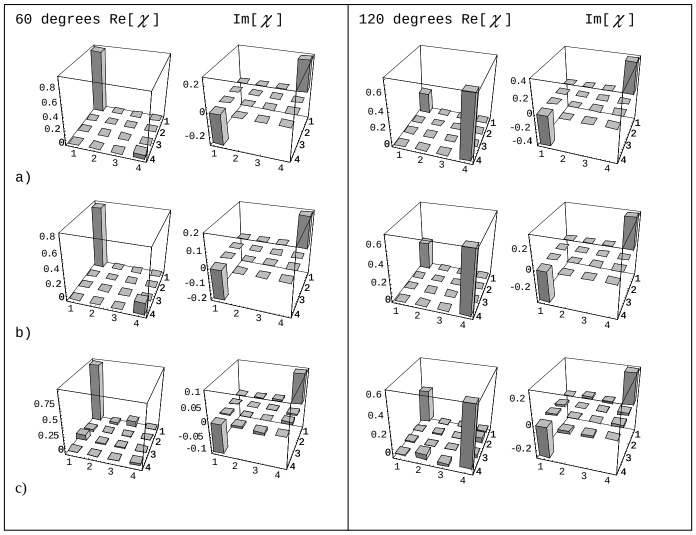

FIG. 5: Quantum process tomography for the remote σ_z rotation of 60° and 120°. We have represented the theoretical values of the χ matrices corresponding to (a) ideal rotation χ_i and (b) a dephased rotation χ_d as well as c) the measured rotation χ_e performed with the set up depicted in Fig. 4. The real parts of the matrix elements of χ are represented in the chart figures on the left while the imaginary ones are represented on the right.

anti-commuting with σ_z (x-rotation) with another HWP at 45° at the output port, which acts as a σ_z operation. The above scheme can also be generalised to implementing controlled operations which commute or anti-commute with σ_z. For example, the control-phase gate commutes with $I \otimes \sigma_z$, $\sigma_z \otimes I$ and $\sigma_z \otimes \sigma_z$. An experiment along these lines, where a non-local CNOT was implemented using linear optics, was reported in [26].

Summarizing, we have implemented a remote rotation by 120° about the z axis on a photonic qubit using shared entanglement and local operations and without performing the rotation directly on the target photons. The whole process was characterized using quantum process tomography and the results agree with the theoretical predictions. The scheme can be generalized to implement remotely any operation belonging to the two classes that allow for protocols different from bidirectional state teleportation [5].

V. CONCLUSIONS

The linearity of quantum mechanics imposes severe constraints on the type of protocols that can be implemented by quantum mechanical means. When combining these constraints with those of locality, one arrives at interesting no-go theorems concerning quantum protocols between spatially separated parties. In this work we have investigated such questions both theoretically and experimentally. In particular, we have considered the possibility of the remote implementation of unitary transformations. If Alice holds a tool that allows for the implementation of a unitary transformation we consider the question whether entanglement and classical communication is sufficient to allow this unitary transformation be applied to an unknown state of a particle held by Bob. We have considered several scenarios and, employing linearity, the non-increase of entanglement under local operations and classical communication (LOCC) as well as the no-signalling condition, several no-go theorems were derived and discussed. We have also identified situations with restricted sets of operations in which the remote application of unitaries can be achieved in a non-trivial way, that is, avoiding state teleportation. Despite their theoretical nature, the results of this work have led to an effort towards the implementation of such protocols and we have described the successful implementation of one of the theoretical protocols that has been developed in this paper.

It is hoped that these results illuminate further how fundamental properties of quantum mechanics impose constraints via linearity, locality and no-signalling but also facilitate new opportunities in the form of generalized measurements and entanglement supplemented by classical communication. The use of multipartite entanglement, as discussed in section III.a, opens up a new front and further applications for quantum communication are foreseeable.

Acknowledgments

The authors would like to thank Claire Bedrock for her patience and her willingness to 'enjoy the challenge' of dealing with a very late submission. This work was supported by the IRC on Quantum Information of the EPSRC (GR/S82176/0), The EU via the Integrated Project QAP, the Thematic Network QUPRODIS (IST-2002-38877), the Leverhulme Trust (F/07058/U), the Chinese National Fundamental Research Program, the NSF of China and the Innovation Funds from the Chinese Academy of Sciences.

[1] A. Peres, *Quantum Theory: Concepts and Methods*, Kluwer, Dordrecht, 1993.(Chapters 5 and 6 and references therein)
[2] M.B. Plenio and V. Vedral, Cont. Phys. 39, 431 (1998); M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000). J. I. Cirac, A. K. Ekert, S. F. Huelga, and C. Macchiavello, Phys. Rev. A **59**,4249 (1999).
[3] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W.K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[4] S.F. Huelga, J.A. Vaccaro, A. Chefles and M.B. Plenio, Phys. Rev. A 6304 (4): 2303 (2001); quant-ph/0005061.
[5] S.F. Huelga, J.A. Vaccaro and M.B. Plenio, Phys. Rev. A **65** 042316 (2002).
[6] M.A. Nielsen and I.L Chuang, Phys. Rev. Lett. **79**, 321 (1997).
[7] See A. Acín, E. Jané and G. Vidal, Phys. Rev. A **64**, 050302(R) (2001) for the problem of directly estimating a unitary transformation.
[8] J. Eisert, K.A. Jacobs, P. Papadopoulos, and M.B. Plenio, Phys. Rev. A **62**, 052317 (2000)
[9] W.K. Wootters and W.H. Zurek, Nature 299, 802 (1982).
[10] G. Vidal, L. Masanes and J.I.Cirac, Phys. Rev. Lett. **88**, 047905 (2002); quant-ph/0102037
[11] M. Hillery, V. Buzek and M. Ziman, Phys. Rev. A **65**, 022301 (2002).
[12] N. Linden and S. Popescu, Phys. Rev. A **64**, 032302 (2001); Chefles, A., C.R. Gilson and S.M. Barnett, Physical Review A, 63, 032314 (2001).
[13] B. Reznik, Y. Aharonov and B. Groisman, Phys. Rev. A **65**, 032312 (2002).
[14] B. Groisman and B. Reznik, Phys. Rev. A **71**, 032322 (2005).
[15] Chui-Ping Yang and Julio Gea-Banacloche, J. Opt. B: Quantum Semiclass. Opt. 3 407-411 (2001).
[16] M. Murao, D. Jonathan, M.B. Plenio and V. Vedral, Phys. Rev. A **59**, 156 (1999)
[17] G.-Y. Xiang, J. Li and G.-C. Guo, Phys. Rev. A **71**, 044304 (2005).
[18] D. F. V. James, P. G. Kwiat, W. J. Munro, and A. G. White, Phys. Rev. A, **64**, 052312 (2001)
[19] P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A. V. Sergienko, and Y. Shih , Phys. Rev. Lett., **75**, 4337 (1995)
[20] P. Hariharan and M. Roy, J. Mod. Opt., **39**, 1811 (1992)
[21] J. Brendel, W. Dultz, and W. Martienssen, Phys. Rev. A, **52**, 2551 (1995)
[22] P. G. Kwiat, E. Waks, A. G. White, I. Appelbaum, and P. H. Eberhard , Phys. Rev. A, **60**, R773 (1999)
[23] J. B. Altepeter, D. Branning, E. Jeffrey, T. C. Wei, P. G. Kwiat, R. T. Thew, J. L. O'Brien, M. A. Nielsen, and A. G. White, Phys. Rev. Lett., **90**, 193601 (2003)

[24] M. D. Bowdrey, et.al., Phys. Lett. A, **294**, 258 (2002)
[25] M. A. Nielsen, Phys. Lett. A, **303**, 249-252 (2002)
[26] Y. F. Huang, X. F. Ren, Y. S. Zhang, L. M. Duan, and G. C. Guo, Phys. Rev. Lett. **93**, 240501 (2004).