Proceedings of the 9th International Conference on Structural Dynamics, EURODYN 2014
Porto, Portugal, 30 June - 2 July 2014
A. Cunha, E. Caetano, P. Ribeiro, G. Müller (eds.)
ISSN: 2311-9020; ISBN: 978-972-752-165-4

# A contribution to the modal identification of the damping factor based on the dissipated energy

Diogo Montalvão[1,2], Júlio M. M. Silva[2]
[1]School of Engineering and Technology, University of Hertfordshire, College Lane, Hatfield AL 10 9 AB, United Kingdom
[2] IDMEC – Instituto de Engenharia Mecânica, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1, 1049-001 Lisboa, Portugal

ABSTRACT: The identification of the modal parameters from frequency response functions is a subject that is not new. However, the starting point often comes from the equations that govern the dynamic motion. In this paper, a novel approach is shown, resulting from an analysis that starts on the dissipated energy per cycle of vibration. For lightly damped systems with conveniently spaced modes, it produced quite accurate results in comparison to the direct application of the method of the inverse, for the numerical examples shown. It also is a simple technique that can be used to produce quick estimates of the modal damping factors. Furthermore, this is also a contribution to further developments on modal analysis and identification methods as, up to today, the developed technique has not yet been proposed.

KEY WORDS: experimental modal analysis (EMA); modal identification; method of the inverse; dissipated energy.

## 1 GENERAL GUIDELINES

Modal identification seeks to obtain the global and local characteristics of vibrating structures using experimental data. This technique may be used either just to obtain the global characteristics (natural frequencies and damping), to directly derive a mathematical model of the structure or to improve a previously built finite element model through what is frequently called updating. The interest of this procedure is acknowledged by the scientific community and many authors have addressed this problem, mainly since the early seventies of the past century [1]. The proposed modal identification procedures cover different levels of sophistication and, in almost all cases, need the use of special software that may not be easy to obtain.

In the past few years, attention has been more focused on Operational Modal Analysis (OMA) rather than in the more traditional Experimental Modal Analysis (EMA). Examples of later developments in OMA identification methods can be found, for instance, in [2-5]. In terms of EMA, later publications are more concerned with Engineering applications, as can be seen, for instance, in [6-7]. OMA deals with operational deflection shapes and many often make use of output-only measurements, this meaning that excitation loads are unknown. EMA makes use of both input forces and output responses in order to determine modal parameters and mode shapes. Numerous modal identification algorithms have been developed in the past thirty years [8]. However, even if in the past recent years not many advances have been seen in terms of EMA modal identification methods, there are still a few interesting results that can be derived.

If the objective is the determination of only the global modal characteristics, it is possible to use simpler approaches producing quick estimates of the desired information. This issue is addressed in this paper where a new simple method is proposed, based on the energy dissipated per cycle of vibration. The proposed methodology showed to be a robust estimator provided the systems under analysis are not heavily damped and the modes are sufficiently separated so that their mutual interference may be assumed as negligible.

This paper presents the proposed new methodology and applies it to numerical examples, showing that it yields reasonably accurate results.

## 2 THEORETICAL DEVELOPMENT

### 2.1 Definitions

The concept of a complex stiffness in vibration problems with viscous or structural (hysteretic) damping is something that has been known for decades. Most often the complex stiffness is defined as the sum of the stiffness itself ($k$, real part) and the damping coefficient ($d$, imaginary part):

$$k^* = k + id \tag{1}$$

To find the real and imaginary parts of the complex stiffness, it is easier if the more conventional viscous damping model is firstly introduced. The well-known second order differential equation of motion - for a single degree of freedom system - is given by:

$$m\ddot{x} + c\dot{x} + kx = Fe^{i\omega t} \tag{2}$$

where $m$ is the mass, $c$ is the viscous damping coefficient, $k$ is the stiffness, $F$ is the amplitude of the oscillatory force and $t$ is the time variable. When excited by an harmonic force at frequency $\omega$, it can easily be proven (and most fundamental texts on vibration theory show it, for instance [1,9]) that for each vibration cycle the system dissipates – through its viscous damper – a quantity of energy directly proportional to the damping coefficient, the excitation frequency and the square of the response amplitude $X$:

$$W_{diss} = \int_0^T f\dot{x}dt = \pi c\omega X^2 \qquad (3)$$

where $T = 2\pi/\omega$ is the time period of oscillation. However, experimental evidence from tests performed on a large variety of materials show that the damping due to internal friction (material hysteresis) is nearly independent of the forcing frequency but still proportional to the square of the response amplitude [10], i. e.:

$$W_{diss} \propto CX^2 \qquad (4)$$

where $C$ is a constant. Therefore, from equations (3) and (4) the equivalent damping coefficient is:

$$c = \frac{C}{\pi\omega} = \frac{d}{\omega} \qquad (5)$$

In such conditions, equation (2) can be re-written as:

$$m\ddot{x} + \frac{d}{\omega}\dot{x} + kx = Fe^{i\omega t} \qquad (6)$$

As $\dot{x} = i\omega x$ for a harmonic vibration, the previous equation may be re-written as:

$$m\ddot{x} + k(1 + i\eta) = Fe^{i\omega t} \qquad (7)$$

where

$$\eta = d/k \qquad (8)$$

is known as the hysteretic damping ratio or damping loss factor. The quantity:

$$k^* = k(1 + i\eta) \qquad (9)$$

is the same complex stiffness as initially described in equation (1).

The latter formulation (7) leads to the conclusion that the dissipated energy per cycle of vibration is independent of the forcing frequency.

### *2.2 A novel approach to the determination of the hysteretic damping coefficient in SDOF systems*

The experimental measurement of the hysteretic damping factor can be carried out by means of cyclic force-displacement tests in the elastic domain [11]. Following the reasoning presented earlier, it is easy to show that the energy dissipated per cycle of oscillation is given by the ellipse area of the force-displacement plot during a complete cycle. Rearrangement of equations (3), (5) and (8) lead to:

$$W_{diss} = \pi dX^2 \qquad (10)$$

This area, the integral of the force along the displacement, corresponds to the non-conservative work done per cycle. In other words, in a plot of force *vs* displacement at a given frequency, damping can be seen as a mechanism that introduces a lag between force and displacement and shows up as an elongated ellipsis [10,11]. In fact, from [12], it can also be shown that the dissipated energy can be written as:

$$W_{diss} = \pi FX \sin\theta \qquad (11)$$

where $\theta$ is the phase angle between the force and the displacement response. From equations (10) and (11) a relationship between the hysteretic damping coefficient, the displacement, the force and the phase angle can be established as:

$$d = \frac{F}{X}\sin(\theta) \qquad (12)$$

For harmonic motion, the ratio between the force and the displacement is a transfer function often referred to as Dynamic Stiffness [1]. Usually, in experimentation, one measures the Receptance instead, which is the inverse of the Dynamic Stiffness:

$$\alpha(\omega) = \frac{x(\omega)}{f(\omega)} \qquad (13)$$

The quantities $x(\omega)$ and $f(\omega)$ are the complex response and force with amplitudes $X(\omega)$ and $F(\omega)$ respectively, both a function of the angular frequency $\omega$. If the amplitude of the receptance is represented by $H(\omega)$, then equation (12) can be re-written more conveniently as:

$$\sin(\theta) = d \cdot H(\omega) \qquad (14)$$

This equation suggests that the hysteretic damping coefficient $d$ can be simply determined from the measurement of the amplitude and phase of the receptance. Once the stiffness is known, equation (8) allows determining the hysteretic damping factor $\eta$.

### *2.3 Determination of the damping factor from the damping coefficient in SDOF systems*

Considering a SDOF system, the receptance (13) may be written as [1]:

$$\alpha(\omega) = \frac{1}{(k - \omega^2 m) + id} \tag{15}$$

If the method of the inverse is applied, one obtains:

$$\frac{1}{\alpha(\omega)} = (k - \omega^2 m) + id \tag{16}$$

Where the imaginary part is:

$$Im\left(\frac{1}{\alpha(\omega)}\right) = d \tag{17}$$

Equation (17) is an alternative form to the equation (14) presented in this paper and will be used for comparison purposes. This equation has been represented in [9] on the Argand plane and a least-squares best fit of a straight line was suggested to be constructed through the data points in order to estimate the damping parameter from the interception of the line with the imaginary axis.

Consider, now, the real part of equation (16):

$$Re\left(\frac{1}{\alpha(\omega)}\right) = k - \omega^2 m \tag{18}$$

This equation (18) is a straight line of the real part of the dynamic stiffness with respect to $\omega^2$, with a negative slope m and the interception of the line with the $y$-axis leads to $k$. Once these values are known, the damping factor can finally be determined from (8) – whether the damping coefficient has been determined by (14) or (17) - and the natural frequency can be estimated from:

$$\omega_n = \sqrt{\frac{k}{m}} \tag{19}$$

### 2.4 Generalisation to MDOF systems

The previous approach is not very useful since most real systems are MDOF, so it must be generalised. It is well known that, in an MDOF, the overall receptance is the sum of each individual DOF contribution:

$$\alpha(\omega) = \sum_r \frac{m_r \bar{A}_r}{k_r - m_r \omega^2 + id_r} \tag{20}$$

where $\bar{A}_r$ is the modal constant for mode $r$ and each mode has its own modal stiffness $k_r$, modal mass $m_r$ and modal damping coefficient $d_r$.

A few simplifications are now convenient. First, consider that the numerator on equation (20) can be assumed as a real quantity. At the vicinity of a resonance, equation (20) is mostly dominated by the corresponding mode and is, approximately:

$$\alpha(\omega_r) \cong \frac{m_r A_r}{k_r - m_r \omega^2 + id_r} + \bar{C}_r \tag{21}$$

in which $\bar{C}_r$ is a complex constant that takes into consideration the influence of all the other modes at the vicinity of mode $r$. Also, consider that the modes are sufficiently spaced and that the receptance is available at points that are far away from nodal lines. In such a case, the influence from other modes is small when compared to the resonant mode and the following approximation can be made:

$$\alpha(\omega_r) \cong \frac{m_r A_r}{k_r - m_r \omega^2 + id_r} \tag{22}$$

Equation (22) resembles the equation of a SDOF with a real modal constant.

If the method of the inverse is, again, applied, and because the simplification that the modal constant is a real number, one obtains:

$$\frac{1}{\alpha(\omega)} = \frac{(k - \omega^2 m)}{m_r A_r} + i\frac{d}{m_r A_r} \tag{23}$$

where the real part is given by:

$$Re\left(\frac{1}{\alpha(\omega)}\right) = \frac{k_r}{m_r A_r} - \omega^2 \frac{m_r}{m_r A_r} \tag{24}$$

The natural frequency can now be determined in a similar away as with (19) and is independent of the modal constant:

$$\omega_r = \sqrt{\frac{k_r/(m_r A_r)}{m_r/(m_r A_r)}} = \sqrt{\frac{k_r}{m_r}} \tag{25}$$

Thus, as long as the modes are sufficiently spaced in frequency and the modal constant is real (or its imaginary part is small in comparison to its real counterpart), this method suggests that the natural frequencies can be determined with a reasonable degree of accuracy.

For the determination of the damping coefficient, the reasoning is similar. Again, it is assumed that an MDOF can be described as the sum of the contribution from several independent SDOFs. In such a case, and taking into consideration (11), the overall dissipated energy at each frequency ω is:

$$\begin{aligned} W_{diss}(\omega) &= \sum_r W_{diss,r}(\omega) \\ &= \sum_r \pi F(\omega) X_r(\omega) \sin(\theta_r(\omega)) \end{aligned} \tag{26}$$

For lightly damped SDOF systems, the phase $\theta(\omega)$ has a value close to zero before the resonance and 180° after the resonance. In any of these cases, $sin(\theta(\omega)) \cong 0$. However, at the resonant frequency the phase $\theta(\omega_r)$ switches from 0° to 180° assuming values close to 90°, which means that, at the resonant frequency, $sin(\theta(\omega_r)) \rightarrow 1$. In other words, near a natural frequency, the dissipated energy (26) assumes a form that resembles the one of a SODF:

$$W_{diss}(\omega) = \pi F(\omega) X_r(\omega)\, sin\big(\theta_r(\omega)\big) + E \tag{27}$$

in which $E$ is a constant that takes into consideration the energy that is being dissipated by other modes. For lightly damped systems $sin(\theta(\omega)) \cong 0$ and $E$ can be assumed close to zero.

This behaviour of $sin(\theta(\omega))$ also suggests that equation (14) can be used to determine the damping coefficient with a certain degree of accuracy in the vicinity of a mode, at least for lightly damped systems and as long as the mode shapes are sufficiently spaced.

## 3 NUMERICAL EXAMPLES

### 3.1 *Numerical Setup*

The performance of equation (14) was compared to equation (17). For a matter of simplicity, these methods are going to be referred as method of the "slope" and method of the "intersection" throughout this paper, respectively. In both cases, the modal hysteretic damping factors were determined using (8). The modal stiffness and modal mass were estimated from a least-squares best fit from equation (24), in which the stiffness parameter is estimated from the interception of the line with the imaginary axis and the modal parameter is estimated from the slope of the line.

A set of different numerical examples were built using equation (20), but covering different scenarios. The different cases are described in table 1:

- Case 1 is a SDOF with real modal constant
- Case 2 is a 2DOF with real modal constants
- Case 3 is a 2DOF with complex modal constants and a heavily damped mode

In the following sections, the results for the identification of the hysteretic damping factors are going to be discussed. In particular, four types of pictures are going to be analysed:

- Plot of the amplitude of the receptance vs frequency
- Plot of $sin(\theta)$ *vs* the amplitude of the receptance (equation (14))
- Plot of the imaginary part *vs* the real part of the the dynamic stiffness

Table 1: Modal properties used in the numerical examples.

| | Mode 1 | | | | Mode 2 | | | |
|---|---|---|---|---|---|---|---|---|
| | Modal Constant 1 | | $f(Hz)$ | $\eta$ | Modal Constant 2 | | $f(Hz)$ | $\eta$ |
| Case | Real | Imag | | | Real | Imag | | |
| 1 | 10e3 | | 20 | 0.05 | - | - | 50 | 0.01 |
| 2 | 10e3 | | 20 | 0.05 | -5e3 | -5e3 | 50 | 0.01 |
| 3 | 10e3 | 10e3 | 20 | 0.5 | -5e3 | -5e3 | 50 | 0.01 |

### 3.2 *Case 1 – SDOF with real modal constant*

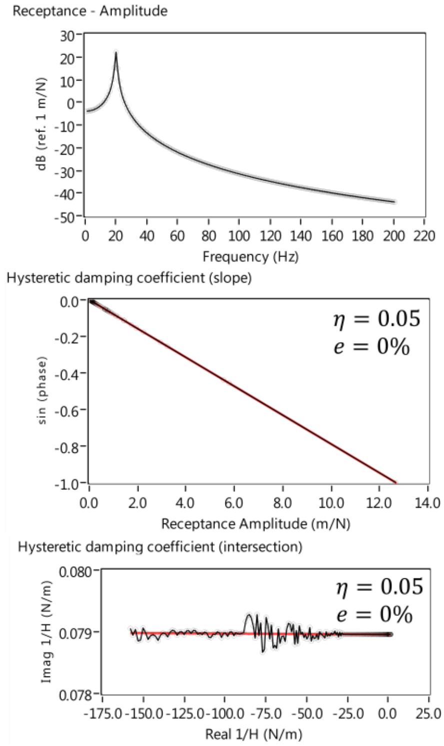


Figure 1: Numerical results of an SDOF with a real modal constant - $f = 20Hz$ and $\eta = 0.05$.

The results for the numerical example of an SDOF with a real modal constant are presented in figure 1. First of all, it can be observed that the plot of $sin(\theta(\omega))$ *vs* the amplitude of the receptance (top right plot) is a straight line intersecting the y-axis at the origin. This suggests that the hysteretic damping coefficient d in equation (14) actually is the slope of this line, which can be estimated constructing a least-squares best fit through the data points. The hysteretic damping factor is then determined from (8).

For this numerical example, both the "slope" and "intersection" methods produced exact solutions, although the method of the "intersection" looks to be slightly sensitive to numerical instability.

### 3.3 *Case 2 – 2DOF with real modal constants*

For a 2 DOF system with real modal constants, the results presented in figure **2** show that it is not possible to make the identification of the modal parameters using the whole frequency span at the same time as it was for a SDOF. This problem is not new and has been circumvented in many other methods by zooming in around the natural frequencies' bandwidth. However, one interesting feature of the method of the "slope" is that the two modes are visible on the upper right corner plot. This is due to the modal constants having opposing signals and, as a consequence, the slopes have opposing signals as well.

Figures 3 and 4 are close-ups at 20Hz and 50Hz, the two resonances respectively. In these two cases the hysteretic damping factor and natural frequency are determined, again, with a very high degree of accuracy (<1% error), regardless of the method used.

It is important to notice, however, that a different number of points was selected for the modal identification from identification in figures 3 to figure 4. One of the reasons is that the bottom-left corner plot in figure 4 is not “as linear” as the corresponding one in figure 3. Also, the top-right corner plot in figure 4 is not “as sharp” as the corresponding one in figure 3. This is because of the influence of the modes in each other. Because of this lack of sharpness in the “slope” method, the identification was carried out centred at the natural frequency (same number of data points to the left and to the right). These suggest that the method of the “slope” is more sensitive to the experience of the user than the method of the “intersection”, as the latter one does not need to be centred at the natural frequency.

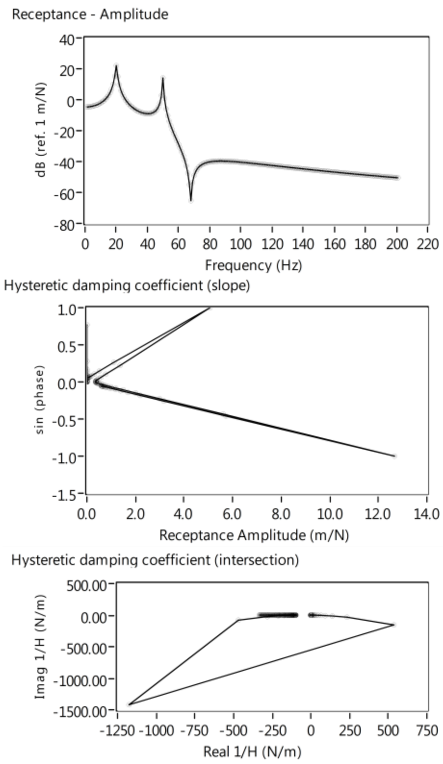


Figure 2: Numerical results of an MDOF with real modal constants - $f_1 = 20Hz$, $f_2 = 50Hz$, $\eta_1 = 0.05$ and $\eta_2 = 0.01$.

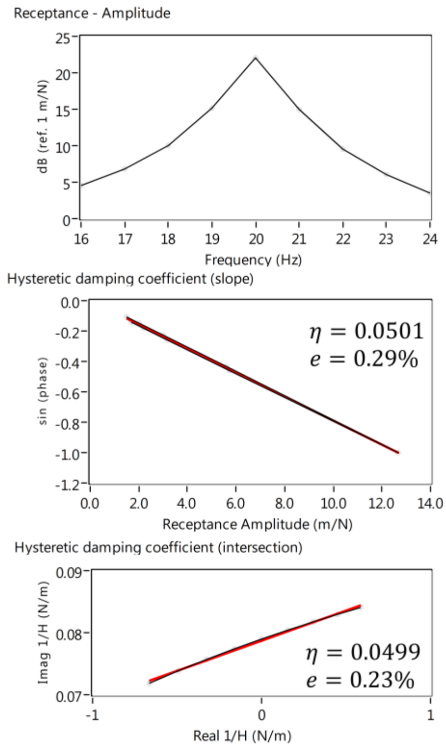


Figure 3: Numerical results of an MDOF with real modal constants, close to the 1st resonance - $f_1 = 20Hz$ and $\eta_1 = 0.05$.

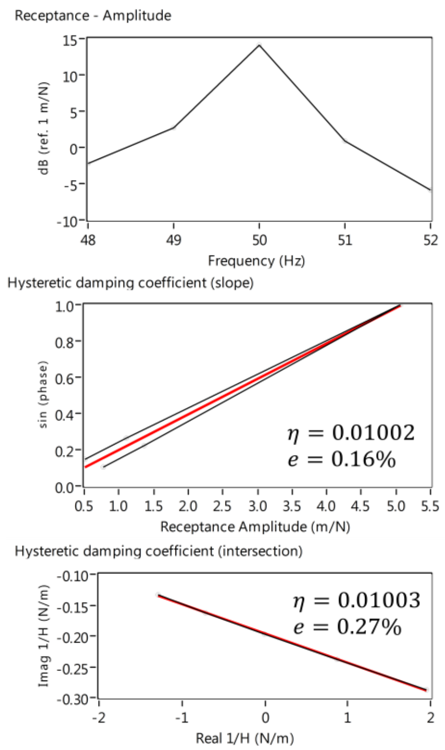


Figure 4: Numerical results of an MDOF with real modal constants, close to the 2nd resonance - $f_2 = 50Hz$ and $\eta_2 = 0.01$.

### 3.4 *Case 3 – 2DOF with complex modal constants and a heavily damped mode*

One of the problems associated to many of the modal identification methods – and the one presented herein is not exempt from this – is that they are mostly effective for lightly damped systems. In this section, an MDOF with complex modal constants and a heavily damped mode (mode 1 at 20Hz with a hysteretic damping factor 10x greater than in the previous sections) is discussed. Figures **Erro! A origem da referência não foi encontrada.** and **Erro! A origem da referência não foi encontrada.** are close-ups at 20Hz and 50Hz, the two resonances respectively. Again, the method of the “slope” produced much better results (10% error) than the method of the “intersection” (93% error). The error was even smaller than the one obtained for the estimate of the 1st mode’s natural frequency (15% error), which typically is the most accurate quantity to determine.

## 4 CONCLUSIONS

A novel method for the identification of the modal damping factor from FRFs was presented. It is based on the dissipated energy per vibration cycle and on the well-known method of the inverse. For lightly damped systems with conveniently spaced modes, it allows determining the modal damping factors in a simple way from the receptance FRFs and with a reasonable degree of accuracy. Due to lack of a better term, it was called method of the “slope” within the context of this paper.

In comparison to the traditional method of the inverse, in which the damping coefficient is determined from the imaginary part of the dynamic stiffness (herein called method of the “intersection”), the method of the “slope” seemed to be slightly sturdier to numerical instability. Also, this method seemed to be much less sensitive to the modal constants, especially when these are complex quantities with large imaginary parts. Both methods work better for lightly damped systems, although the method of the “slope”, again, seemed to perform better with more heavily damped systems. However, in terms of limitations, the method of the “slope” is more sensitive to the experience of the user than the method of the “intersection”, because the number of points chosen to the right or to the left may have a strong influence on the quality of the identification, whereas for the method of the “intersection” this is not as relevant.

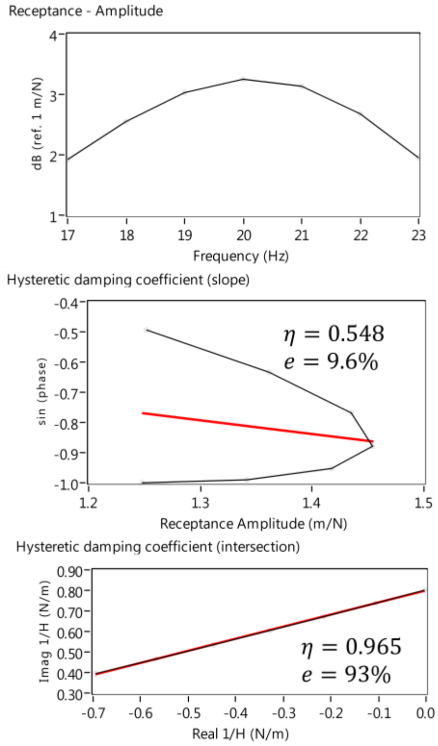


Figure 5: Numerical results of an MDOF with complex modal constants, close to the 1st resonance (heavily damped) - $f_1 = 20Hz$ and $\eta_1 = 0.5$.

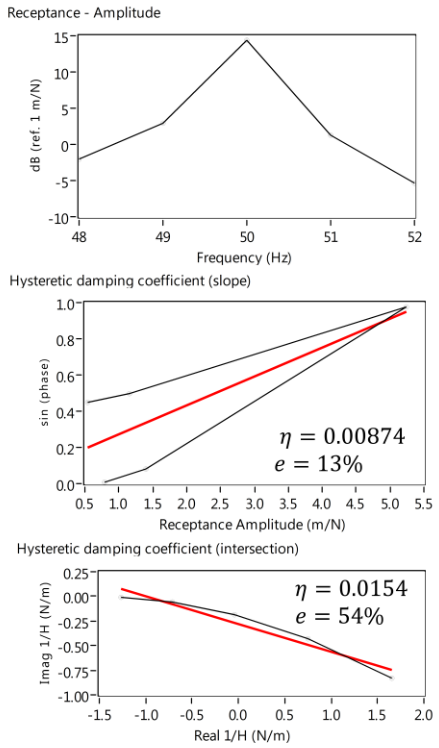


Figure 6: Numerical results of an MDOF with complex modal constants, close to the 2nd resonance - $f_2 = 50Hz$ and $\eta_2 = 0.01$.

## REFERENCES


[1] N.M.M. Maia, J.M.M. Silva, Theoretical and Experimental Modal Analysis, Research Studies Press and John Wiley and Sons, Somerset, 1997.

[2] A. Sadhua, B. Hazraa, S. Narasimhana. Decentralized modal identification of structures using parallel factor decomposition and sparse blind source separation, Mech. Syst. Signal Process., 41 (2006) 396-419.

[3] A. Siu-Kui, Uncertainty law in ambient modal identification—Part I: Theory, Mech. Syst. Signal Process., In Press, Corrected Proof, Available online 31 October 2013

[4] L. Thien-Phu, P. Paultre, Modal identification based on the time–frequency domain decomposition of unknown-input dynamic tests, Int. J. Mech. Sci, 71 (2013) 41-50.

[5] L. Cheng, D. Zheng, The identification of a dam's modal parameters under random support excitation based on the Hankel matrix joint approximate diagonalization technique, Mech. Syst. Signal Process. 42 (2014) 42-57.

[6] J.L. Zapico-Valle, M. García-Diéguez, R. Alonso-Camblor, Nonlinear modal identification of a steel frame, Eng Struct, 56 (2013) 246-259.

[7] J.A. Gonilha, J.R. Correia, F.A. Branco, E. Caetano, A. Cunha, Modal identification of a GFRP-concrete hybrid footbridge prototype: Experimental tests and analytical and numerical simulations, Compos. Struct., 106 (2013) 724-733.

[8] L. Zhang, R. Brincker, P. Andersen, An Overview of Operational Modal Analysis: Major Development and Issues, Proceedings of the 1st International Operational Modal Analysis Conference (IOMAC), April 26-27, Copenhagen, Denmark, 2005.

[9] D.J. Ewins, Modal Testing: Theory and Practice, Research Studies Press And John Wiley and Sons, Letchworth, 1984.

[10] B.J. Lazan, Damping of materials and members in structural mechanics. Pergamon Press, Oxford, 1968.

[11] D. Montalvão, A.M.R. Ribeiro, J.D. Silva, R.A. Cláudio, Experimental Measurement of the complex Young's modulus on a CFRP laminate considering the constant hysteretic damping model, Compos. Struct., 97 (2013) 91-98.

[12] L. Meirovitch, Elements of Vibration Analysis, McGraw-Hill, International Edition, 1986.

[13] D. Montalvão, A Modal-based Contribution to Damage Location in Laminated Composite Plates, PhD thesis, Instituto Superior Técnico, Technical University of Lisbon, Portugal, 2010.

[14] J.M.M. Silva, N.M.M. Maia, A.M.R. Ribeiro, Structural Dynamic Identification with Modal Constant Consistency using the Characteristic Response Function (CRF), Machine Vibration, 5 (1996) 83-88.