

On classical q -deformations of integrable σ -models

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ABSTRACT: A procedure is developed for constructing deformations of integrable σ -models which are themselves classically integrable. When applied to the principal chiral model on any compact Lie group F , one recovers the Yang-Baxter σ -model introduced a few years ago by C. Klimčík. In the case of the symmetric space σ -model on F/G we obtain a new one-parameter family of integrable σ -models. The actions of these models correspond to a deformation of the target space geometry and include a torsion term. An interesting feature of the construction is the q -deformation of the symmetry corresponding to left multiplication in the original models, which becomes replaced by a classical q -deformed Poisson-Hopf algebra. Another noteworthy aspect of the deformation in the coset σ -model case is that it interpolates between a compact and a non-compact symmetric space. This is exemplified in the case of the $SU(2)/U(1)$ coset σ -model which interpolates all the way to the $SU(1,1)/U(1)$ coset σ -model.

KEYWORDS: Integrable Field Theories, Sigma Models, Quantum Groups

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1 Introduction

The property of integrability is extremely scarce among two-dimensional σ -models. And yet when present it provides a powerful tool in the study of various exact properties of these models. There is, however, no systematic way of proving whether or not a two-dimensional σ -model is integrable. In light of this, an interesting question to consider is the following: given an integrable σ -model, is it possible to construct a deformation of this model which is itself integrable?

In the case of the $SU(2)$ principal chiral model, an example of such a deformation is given by the diagonal anisotropic $SU(2)$ principal chiral model introduced by Cherednik in [1]. The action for the $SU(2)$ -valued field g of this model may be written as

$$S_C[g] = -\frac{1}{2} \int d\tau d\sigma \operatorname{tr} \left(\operatorname{ad}(\partial_+ g g^{-1}) J \operatorname{ad}(\partial_- g g^{-1}) \right),$$

where $J = \operatorname{diag}(J_1, J_2, J_3)$ is a diagonal matrix, the effect of which is to deform the metric away from the Killing form of $\mathfrak{su}(2)$. This model is known to be integrable [1] and provides a two-parameter deformation of the principal chiral model.

In the special case $J_1 = J_2 \neq J_3$ it reduces to the squashed sphere σ -model, where the parameter $C = J_3/J_1$ describes the squashing of the 3-sphere. As a result of this squashing when $C \neq 1$, the global $SU(2)_L \times SU(2)_R$ symmetry of the principal chiral model is broken down to $SU(2)_L \times U(1)_R$. However, it was recently argued in [2, 3] that a certain deformation of the $SU(2)_R$ symmetry is still realised in the squashed sphere σ -model. Specifically, as the deformation is turned on, the $SU(2)_R$ symmetry gets replaced by a classical q -deformed $U_q^{\mathcal{P}}(\mathfrak{sl}_2)$ symmetry, where the algebraic deformation parameter $q = q(C)$ is a function of the geometric squashing parameter C .

A generalisation of the above one-parameter deformation for the principal chiral model on any compact Lie group F is the so called Yang-Baxter σ -model introduced by Klimčík in [4]. In a subsequent paper it was then proved that this deformation is in fact also integrable [5]. Using the conventions of the present paper, the action of this model reads

$$S_K[g] = -\frac{1}{2} \int d\tau d\sigma \kappa \left(\partial_+ g g^{-1}, \frac{(1 + \eta^2)^2}{1 - \eta R} \partial_- g g^{-1} \right),$$

where κ is the Killing form of the Lie algebra $\mathfrak{f} = \operatorname{Lie}(F)$ and $\eta \geq 0$ is the deformation parameter. Here R is a certain solution of the modified classical Yang-Baxter equation on \mathfrak{f} . In the limit $\eta \rightarrow 0$ this action reduces to that of the principal chiral model. Furthermore, in the case $F = SU(2)$ it reduces to the action of the diagonal anisotropic $SU(2)$ principal chiral model with $J_1 = J_2 \neq J_3$.

The first objective of this paper is to put forward a procedure for deforming integrable σ -models in a way which manifestly preserves their integrability. The cases that we shall consider here are the principal chiral model on any compact Lie group F and the coset σ -model on a symmetric space F/G . The second objective is to show that the models so obtained admit a classical q -deformed symmetry.

In the case of the principal chiral model, we shall in fact recover in this way the Yang-Baxter σ -model. Its integrability will, however, be automatic from our construction. Furthermore, working in the Hamiltonian formalism will also enable us to show that the Yang-Baxter σ -model admits a classical q -deformed $U_q^{\mathcal{P}}(\mathfrak{f}) \times F_R$ symmetry, where $q = q(\eta)$ is a certain function of the deformation parameter η . In the limit $\eta \rightarrow 0$ this reduces to the global $F_L \times F_R$ symmetry of the principal chiral model. This feature of the Yang-Baxter σ -model therefore generalises the analogous q -deformation exhibited in [2, 3] for the symmetries of the squashed sphere σ -model.

Most importantly, our procedure admits a straightforward generalisation to coset σ -models. We shall indeed construct a new one-parameter deformation of the coset σ -model

on F/G where F is a compact Lie group and $G = \exp \mathfrak{g}$ is the Lie group associated with the subalgebra \mathfrak{g} of \mathfrak{f} fixed by an order 2 automorphism $\sigma : \mathfrak{f} \rightarrow \mathfrak{f}$. The resulting action takes the form

$$S[g] = -\frac{1}{2} \int d\tau d\sigma \kappa \left((g^{-1} \partial_+ g)^{(1)}, \frac{1 + \eta^2}{1 - \eta R_g \circ P_1} (g^{-1} \partial_- g)^{(1)} \right),$$

where $R_g = \text{Ad } g^{-1} \circ R \circ \text{Ad } g$ and $P_1 M = M^{(1)}$ is the projection of $M \in \mathfrak{f}$ onto the subspace of \mathfrak{f} on which the automorphism σ has eigenvalue -1 . Just as in the case of the Yang-Baxter σ -model, we will show that this model also admits a q -deformed $U_q^{\mathcal{P}}(\mathfrak{f})$ symmetry where $q = q(\eta)$ is again a function of the real deformation parameter η .

Our strategy for deforming the principal chiral model and coset σ -models crucially exploits the existence of a second Poisson bracket compatible with the original one. Such a compatible bracket was introduced in [6] for the $SU(2)$ principal chiral model and this was subsequently generalised to all other principal chiral models and coset σ -models in [7]. Recall that the integrability of these models at the Hamiltonian level follows from the Poisson bracket of their Lax matrix taking the specific form in [8, 9]. In order to construct an integrable deformation we should therefore ensure that this latter property is preserved. Now in both models, the Lax matrix depends on the canonical fields only indirectly through certain currents. We shall not modify this dependence of the Lax matrix on these currents. Instead, what we shall deform is the way these currents depend on the underlying canonical fields. This will be achieved by deforming the Poisson bracket of the currents, which we do by adding a multiple of the compatible Poisson bracket. As a result, the Hamilton dynamics of the canonical fields will be deformed. After taking the inverse Legendre transform this procedure leads to the above Lagrangians for the deformed models.

This article is organised as follows. The procedure is first presented in the case of the principal chiral model in section 2. After recalling some well known properties relating to the integrability and symmetries of this model, we introduce the deformed Poisson bracket in subsection 2.2. The resulting deformation of the relation between the Lax matrix and canonical variables is worked out in subsections 2.3 and 2.4. The deformation of the global $F_L \times F_R$ symmetry is studied in the next subsection. We end this section by deriving the action describing our deformed model, thereby showing that it coincides with the Yang-Baxter σ -model. Section 3 is devoted to the deformation of symmetric space σ -models. We follow exactly the same steps as for the principal chiral model. The corresponding action is computed in subsection 3.5. In section 4 we study the simplest example of the deformed $SU(2)/U(1)$ coset σ -model. It provides an interesting interpolation between coset σ -models on the compact and non-compact symmetric spaces $SU(2)/U(1)$ and $SU(1,1)/U(1)$, respectively. This article includes four appendices. Some notations on compact real Lie algebras and a reminder on the Iwasawa decomposition are found in appendix A. Details for the proof of the q -Poisson-Serre relations are given in appendix B. Finally, the last two appendices are respectively devoted to a discussion of the modified classical Yang-Baxter equation and the deformed Poisson bracket used in the case of the coset σ -models.

2 Deforming the principal chiral model

2.1 Principal chiral model

We begin this section by reviewing aspects of the principal chiral model on a compact Lie group F which will be relevant for our purposes. Although these are standard properties, it is important to recall them in order to emphasise those features of the model which we shall deform later.

Hamiltonian, equations of motion and Lax matrix. The principal chiral model may be described by a pair of fields $j_0(\sigma)$ and $j_1(\sigma)$ each of which takes values in the compact Lie algebra $\mathfrak{f} = \text{Lie}(F)$. We shall consider the case where the underlying space, parameterised by σ , is the entire real line. In particular, the fields $j_0(\sigma)$ and $j_1(\sigma)$ will be assumed to decay sufficiently rapidly at infinity. Their Poisson brackets are given by

$$\{j_{0\mathbf{1}}(\sigma), j_{0\mathbf{2}}(\sigma')\} = -[C_{\mathbf{12}}, j_{0\mathbf{2}}(\sigma)]\delta_{\sigma\sigma'} \quad (2.1a)$$

$$\{j_{0\mathbf{1}}(\sigma), j_{1\mathbf{2}}(\sigma')\} = -[C_{\mathbf{12}}, j_{1\mathbf{2}}(\sigma)]\delta_{\sigma\sigma'} + C_{\mathbf{12}}\delta'_{\sigma\sigma'} \quad (2.1b)$$

$$\{j_{1\mathbf{1}}(\sigma), j_{1\mathbf{2}}(\sigma')\} = 0. \quad (2.1c)$$

We denote by $C_{\mathbf{12}} = \kappa_{ab}T^a \otimes T^b$ the tensor Casimir with κ_{ab} the components of the inverse of the Killing form κ on \mathfrak{f} in any basis T^a (see appendix A for notations).

The Hamiltonian of the model reads

$$H_{\text{PCM}} = -\frac{1}{2} \int_{-\infty}^{+\infty} d\sigma (\kappa(j_0, j_0) + \kappa(j_1, j_1)). \quad (2.2)$$

The resulting equations of motion, with $\partial_\tau = \{H_{\text{PCM}}, \cdot\}$, take the form of the conservation equation and the zero curvature equation

$$-\partial_\tau j_0 + \partial_\sigma j_1 = 0, \quad (2.3a)$$

$$\partial_\tau j_1 - \partial_\sigma j_0 - [j_0, j_1] = 0. \quad (2.3b)$$

The integrability of these equations of motion is encoded in the usual Lax matrix

$$\mathcal{L}(\lambda) = \frac{1}{1 - \lambda^2} (j_1 + \lambda j_0), \quad (2.4)$$

which takes values in the loop algebra $\widehat{\mathfrak{f}} = \mathfrak{f} \otimes \mathbb{C}((\lambda))$.

Symmetry algebra and group valued field. It is instructive to recall some properties of the global $F_L \times F_R$ symmetry of the principal chiral model. Indeed, part of these symmetries will turn out to be deformed in the model we shall construct.

It is immediate from equation (2.3a) that $Q^R = \int d\sigma j_0$ is a conserved quantity. By introducing the group valued principal chiral field $g \in F$ through the relation $j_1 = -g^{-1}\partial_\sigma g$, this charge is seen to generate the F_R symmetry of the model acting as $g \mapsto gU_R$. Indeed, the Poisson brackets (2.1b) and (2.1c) lifted to the field g read

$$\{j_{0\mathbf{1}}(\sigma), g_{\mathbf{2}}(\sigma')\} = g_{\mathbf{2}}(\sigma)C_{\mathbf{12}}\delta_{\sigma\sigma'}, \quad (2.5a)$$

$$\{g_{\mathbf{1}}(\sigma), g_{\mathbf{2}}(\sigma')\} = 0. \quad (2.5b)$$

Furthermore, this charge Q^R appears at order λ^{-1} in the expansion of the monodromy matrix at $\lambda = \infty$ since the expansion of the Lax matrix (2.4) there begins with

$$\mathcal{L}(\lambda) = -\lambda^{-1}j_0 + O(\lambda^{-2}). \tag{2.6}$$

It turns out that both the field g and the F_L symmetry, acting as $g \mapsto U_L g$, may be conveniently described in terms of the leading behaviour of the Lax matrix at the point $\lambda = 0$. By virtue of the definition of j_1 in terms of g , the value of the Lax matrix at $\lambda = 0$ is $\mathcal{L}(0) = -g^{-1}\partial_\sigma g$. This shows that the field $g \in F$ may be characterised rather abstractly as the gauge transformation parameter which sends $\mathcal{L}(0)$ to zero. The generator of the F_L symmetry can then be extracted from the next order in the expansion of the gauge transformed Lax matrix at $\lambda = 0$. Indeed, if we define $l_0 = g j_0 g^{-1}$ we have

$$\mathcal{L}^g(\lambda) := \partial_\sigma g g^{-1} + g \mathcal{L}(\lambda) g^{-1} = \lambda l_0 + O(\lambda^2). \tag{2.7}$$

Furthermore, the definition of l_0 and the Poisson brackets (2.1a) and (2.5) lead to

$$\{l_{0\mathbf{1}}(\sigma), l_{0\mathbf{2}}(\sigma')\} = [C_{\mathbf{1}\mathbf{2}}, l_{0\mathbf{2}}(\sigma)] \delta_{\sigma\sigma'}, \tag{2.8a}$$

$$\{l_{0\mathbf{1}}(\sigma), g_{\mathbf{2}}(\sigma')\} = C_{\mathbf{1}\mathbf{2}} g_{\mathbf{2}}(\sigma) \delta_{\sigma\sigma'}. \tag{2.8b}$$

It therefore follows that the generator of the F_L symmetry is $Q^L = \int d\sigma l_0$ and moreover it appears as the coefficient of λ in the expansion at $\lambda = 0$ of the gauge transformed monodromy matrix.

Let us briefly summarise the above by remarking that the pair of fields g and l_0 may roughly speaking be regarded as canonical fields for the principal chiral model with Poisson brackets given in (2.5b) and (2.8). The pair (g, l_0) takes values in the canonical right trivialisation of the cotangent bundle of F . Moreover, both these fields may be extracted from the Lax matrix using the following scheme:

- The field g is characterised by the condition $\mathcal{L}^g(0) = 0$ which fixes $j_1 = -g^{-1}\partial_\sigma g$.
- The field l_0 is obtained as $\frac{\partial \mathcal{L}^g}{\partial \lambda}(0) = l_0$, implying the relation $j_0 = g^{-1}l_0 g$.

2.2 Setting up the deformation

Deformed Poisson bracket. Our starting point for constructing a deformation of the principal chiral model in the Hamiltonian formalism will be to deform its Poisson bracket. A natural way to do this is to combine the original Poisson bracket $\{\cdot, \cdot\}$ of the current in (2.1) with a compatible Poisson bracket, say $\{\cdot, \cdot\}'$. In the case at hand there is a natural candidate for $\{\cdot, \cdot\}'$, namely the Poisson bracket associated with the Faddeev-Reshetikhin model [6]. Indeed, its compatibility with (2.1) was shown in [7]. We therefore consider the following linear combination of Poisson brackets

$$\{\cdot, \cdot\}_\epsilon := \{\cdot, \cdot\} + \epsilon^2 \{\cdot, \cdot\}', \tag{2.9}$$

where the parameter ϵ is taken to be real and positive. When $\epsilon = 0$ this bracket corresponds to the original undeformed Poisson bracket $\{\cdot, \cdot\}_0 = \{\cdot, \cdot\}$ whereas when ϵ tends to infinity

it becomes proportional to the Faddeev-Reshetikhin bracket $\{\cdot, \cdot\}'$. For any other value $\epsilon > 0$ it reads

$$\{j_{0\underline{1}}(\sigma), j_{0\underline{2}}(\sigma')\}_\epsilon = -(1 - \epsilon^2)[C_{\underline{12}}, j_{0\underline{2}}(\sigma)]\delta_{\sigma\sigma'}, \tag{2.10a}$$

$$\{j_{0\underline{1}}(\sigma), j_{1\underline{2}}(\sigma')\}_\epsilon = -(1 - \epsilon^2)[C_{\underline{12}}, j_{1\underline{2}}(\sigma)]\delta_{\sigma\sigma'} + C_{\underline{12}}\delta'_{\sigma\sigma'}, \tag{2.10b}$$

$$\{j_{1\underline{1}}(\sigma), j_{1\underline{2}}(\sigma')\}_\epsilon = \epsilon^2[C_{\underline{12}}, j_{0\underline{2}}(\sigma)]\delta_{\sigma\sigma'}. \tag{2.10c}$$

Lax matrix and Hamiltonian. In order to ensure that the deformed model remains integrable as we vary the deformation parameter ϵ , we shall do two things.

On the one hand, and in the spirit of [7], we shall require that the Lax matrix of the deformed model be the same function of j_0 and j_1 , independent of ϵ . In other words, we will take the same Lax matrix (2.4) for every value of the parameter ϵ .

On the other hand, we shall also insist that the dynamics of the fields (j_0, j_1) remain the same as we vary ϵ . Nevertheless, since the Poisson brackets of (j_0, j_1) do depend on ϵ , this implies that the dependence of (j_0, j_1) on the canonical fields will vary with ϵ . Consequently, the dynamics of these canonical fields will be deformed. When ϵ vanishes, the principal chiral field g itself together with the field j_0 , or equivalently l_0 , may be regarded as canonical fields in view of (2.5b) and (2.8). The possibility to deform the principal chiral model will therefore come from the freedom in defining the field g at non-zero values of the deformation parameter ϵ . We shall come back in detail to this important point in section 2.3 below.

The Lax matrix (2.4) depends linearly on the fields (j_0, j_1) . Therefore, in order to find the Hamiltonian H^ϵ which generates the same dynamics on these fields as the principal chiral model but with respect to the deformed Poisson bracket (2.10), we should solve the following equation

$$\{H^\epsilon, \mathcal{L}\}_\epsilon = \{H_{\text{PCM}}, \mathcal{L}\}. \tag{2.11}$$

By using the fact that the Hamiltonian H_{PCM} has vanishing Faddeev-Reshetikhin Poisson bracket with any function of (j_0, j_1) , it is easy to see that $H^\epsilon = H_{\text{PCM}}$ is also the Hamiltonian with respect to the deformed bracket.

Deformed twist function. In view of deforming the definition of the principal chiral field g as given in section 2.1, we first need to understand the distinguishing characteristic of the special point $\lambda = 0$ entering this definition.

In the Hamiltonian formalism, the algebraic ingredients underpinning the integrability of non-ultralocal models of interest in this paper were emphasised in [7], to which the reader is referred. Aside from the loop algebra $\widehat{\mathfrak{f}}$ and the Lax matrix $\mathcal{L}(\lambda)$ valued in $\widehat{\mathfrak{f}}$, an essential role is played by the standard split R -matrix \mathcal{R} , which is a solution of the modified classical Yang-Baxter equation on $\widehat{\mathfrak{f}}$ (see appendix C). An equally important ingredient in this setup is the twist function $\varphi(\lambda)$. As explained in [7], in this language the Poisson bracket of any two functions of the Lax matrix may be expressed in terms of the rational inner product on $\widehat{\mathfrak{f}}$ and the twisted R -matrix $\mathcal{R} \circ \tilde{\varphi}^{-1}$, where $\tilde{\varphi}$ denotes multiplication by the twist function $\varphi(\lambda)$. The twist functions of the principal chiral model and the Faddeev-Reshetikhin model

are given respectively by

$$\varphi_{\text{PCM}}(\lambda) = -1 + \frac{1}{\lambda^2}, \quad \varphi_{\text{FR}}(\lambda) = 1. \tag{2.12}$$

Note that in this formalism, the compatibility between the Poisson brackets of these two models may be inferred from [10].

The Poisson bracket $\{f_1, f_2\}_\epsilon(\mathcal{L})$ of any two functions f_1 and f_2 can be computed in two ways. By definition, it is given by the linear combination of the brackets $\{f_1, f_2\}(\mathcal{L})$ and $\{f_1, f_2\}'(\mathcal{L})$ which are respectively linear in $\mathcal{R} \circ \tilde{\varphi}_{\text{PCM}}^{-1}$ and $\mathcal{R} \circ \tilde{\varphi}_{\text{FR}}^{-1}$. Alternatively, one can determine the twist function φ_ϵ for the deformed Poisson bracket (2.9) and then compute $\{f_1, f_2\}_\epsilon(\mathcal{L})$ directly in terms of $\mathcal{R}_\epsilon := \mathcal{R} \circ \tilde{\varphi}_\epsilon^{-1}$. Restricting to linear functions f_1 and f_2 of \mathcal{L} , one has

$$\{\mathcal{L}_1(\sigma), \mathcal{L}_2(\sigma')\}_\epsilon = [\mathcal{R}_{\epsilon\mathbf{12}}, \mathcal{L}_1(\sigma)]\delta_{\sigma\sigma'} - [\mathcal{R}_{\epsilon\mathbf{12}}^*, \mathcal{L}_2(\sigma)]\delta_{\sigma\sigma'} + (\mathcal{R}_{\epsilon\mathbf{12}} + \mathcal{R}_{\epsilon\mathbf{12}}^*)\delta'_{\sigma\sigma'}$$

where $\mathcal{R}_{\epsilon\mathbf{12}}$ and $\mathcal{R}_{\epsilon\mathbf{12}}^*$ are respectively the kernels of \mathcal{R}_ϵ and its adjoint with respect to the rational inner product on $\hat{\mathfrak{f}}$. The reader is referred to [7] for details.

Putting all this together we obtain a simple expression for the inverse of the twist function of the deformed Poisson bracket

$$\varphi_\epsilon(\lambda)^{-1} = \varphi_{\text{PCM}}(\lambda)^{-1} + \epsilon^2 \varphi_{\text{FR}}(\lambda)^{-1}.$$

Substituting the definitions (2.12) we find the deformed twist function to be

$$\varphi_\epsilon(\lambda) = \frac{1 - \lambda^2}{(1 - \epsilon^2)\lambda^2 + \epsilon^2}. \tag{2.13}$$

Poles of the deformed twist function. It is clear from (2.12) that the point $\lambda = 0$, from which the principal chiral field g may be extracted, corresponds in fact to the pole of the twist function $\varphi_{\text{PCM}}(\lambda)$. It is therefore natural to expect that the poles of the deformed twist function (2.13) will be of particular importance in defining the group valued field corresponding to the deformed theory. Moreover, the symmetry generators of the deformed model will be obtained by expanding the monodromy matrix around these points. They are located at

$$\lambda_\pm = \pm \frac{i\epsilon}{\sqrt{1 - \epsilon^2}} \tag{2.14}$$

and have the property $\lambda_- = \bar{\lambda}_+$ which we will make use of later. Hence, the double pole at $\lambda = 0$ of the twist function $\varphi_{\text{PCM}}(\lambda)$ is seen to split into a pair of single poles as we turn on the deformation parameter ϵ . Another interesting feature of (2.14) is that the poles move off to infinity as $\epsilon \rightarrow 1$.

2.3 Defining the group valued field

Definition of g . Mimicking the interpretation of the principal chiral field as the parameter of a gauge transformation sending the Lax matrix $\mathcal{L}(0)$ to zero, we would like to define the field g for $\epsilon \neq 0$ as the parameter of a gauge transformation of some sort. However,

since for $\epsilon \neq 0$ there are now two poles at λ_{\pm} , we should consider both Lax matrices $\mathcal{L}(\lambda_+)$ and $\mathcal{L}(\lambda_-)$.

Consider first $\mathcal{L}(\lambda_+)$. Since we want the field g to belong to the compact Lie group F for any ϵ , i.e. $g^\dagger = g^{-1}$, we should ensure that $\partial_\sigma g g^{-1}$ takes values in \mathfrak{f} . We therefore define g so that $\partial_\sigma g g^{-1}$ coincides with the component along \mathfrak{f} in the Iwasawa decomposition (A.7) of $-g\mathcal{L}(\lambda_+)g^{-1}$. In other words, we define the field $g \in F$ as the parameter of a gauge transformation such that

$$\mathcal{L}^g(\lambda_+) = \partial_\sigma g g^{-1} + g\mathcal{L}(\lambda_+)g^{-1} \tag{2.15a}$$

belongs to $\mathfrak{h}_0 \oplus \mathfrak{n}^+ \subset \mathfrak{b}^+$, where $\mathfrak{h}_0, \mathfrak{n}^+$ and \mathfrak{b}^+ are defined in appendix A. Consider now the effect of this gauge transformation at the other point $\lambda = \lambda_-$, namely

$$\mathcal{L}^g(\lambda_-) = \partial_\sigma g g^{-1} + g\mathcal{L}(\lambda_-)g^{-1}. \tag{2.15b}$$

Since the fields j_0 and j_1 both take values in \mathfrak{f} we have $j_a^\dagger = -j_a$ for $a = 0, 1$, from which the reality condition on the Lax matrix follows

$$\mathcal{L}(\lambda)^\dagger = -\mathcal{L}(\bar{\lambda}). \tag{2.16}$$

In particular this means that $\mathcal{L}(\lambda_+)^\dagger = -\mathcal{L}(\lambda_-)$ which combined with (2.15) yields

$$\mathcal{L}^g(\lambda_-) = -\mathcal{L}^g(\lambda_+)^\dagger. \tag{2.17}$$

This implies, firstly, that $\mathcal{L}^g(\lambda_-)$ belongs to the lower Borel subalgebra \mathfrak{b}^- of $\mathfrak{f}^{\mathbb{C}}$, or more precisely to $\mathfrak{h}_0 \oplus \mathfrak{n}^-$. Secondly, since the restriction of $\mathcal{L}^g(\lambda_+)$ to the Cartan subalgebra \mathfrak{h} of $\mathfrak{f}^{\mathbb{C}}$ is actually contained in \mathfrak{h}_0 , we have $\mathcal{L}^g(\lambda_-)|_{\mathfrak{h}} = -\mathcal{L}^g(\lambda_+)|_{\mathfrak{h}}$.

Therefore, by a single gauge transformation with parameter g we can ensure that the gauge transformed Lax matrix defined as $\mathcal{L}^g(\lambda) = \partial_\sigma g g^{-1} + g\mathcal{L}(\lambda)g^{-1}$ has the property that

$$\begin{aligned} \text{(i)} \quad & \mathcal{L}^g(\lambda_{\pm}) \in \mathfrak{b}^{\pm}, \\ \text{(ii)} \quad & \mathcal{L}^g(\lambda_-)|_{\mathfrak{h}} = -\mathcal{L}^g(\lambda_+)|_{\mathfrak{h}}. \end{aligned} \tag{2.18}$$

To see why this definition of g is a deformation of the principal chiral field, consider the limit when $\epsilon \rightarrow 0$. In this limit, the pair of points λ_{\pm} in (2.14) degenerate to a single point at $\lambda = 0$. Property (i) then requires that $\mathcal{L}^g(0)$ be in both \mathfrak{b}^+ and \mathfrak{b}^- and hence $\mathcal{L}^g(0) \in \mathfrak{h}$. But then property (ii) implies that $\mathcal{L}^g(0) = 0$, which is exactly the defining property of the principal chiral field.

Singularity at $\epsilon = 1$. When the deformation parameter lies in the range $0 < \epsilon < 1$, the points λ_{\pm} defined in (2.14) are distinct and the above procedure can be used to define the field g . As explained above, when $\epsilon = 0$ the pair of points λ_{\pm} merge at $\lambda = 0$ and g becomes identified with the principal chiral field. Likewise, as $\epsilon \rightarrow 1$ the pair of points λ_{\pm} both move off towards infinity. However, the difference here is that in the limit $\lambda \rightarrow \infty$ the Lax matrix vanishes identically and the above procedure for defining g no longer makes sense. As we shall see later, this is a symptom of the fact that the deformed theory is only defined for $0 \leq \epsilon < 1$.

Definition of the conjugate momentum. So far we have defined a field g for any value of the deformation parameter ϵ in the range $0 \leq \epsilon < 1$, which identifies in the limit $\epsilon \rightarrow 0$ with the principal chiral field. In order to describe the dynamics of this new field g we shall need to relate it to the components (j_0, j_1) of the current whose dynamics is known, and in fact independent of ϵ . In analogy with the Hamiltonian analysis of the principal chiral model, this requires introducing another field X which will essentially turn out to be the conjugate momentum of g . We will then be able to express (j_0, j_1) in terms of the pair of Hamiltonian fields (g, X) .

We therefore define

$$X = \frac{i}{2\gamma} (\mathcal{L}^g(\lambda_+) - \mathcal{L}^g(\lambda_-)), \tag{2.19}$$

where the parameter γ is a normalisation to be fixed later. In the limit $\epsilon \rightarrow 0$, this expression has to identify with the derivative of $\mathcal{L}^g(\lambda)$ in λ evaluated at $\lambda = 0$. In view of (2.14) this fixes the leading behaviour of γ to be $\gamma \sim -\epsilon$ as $\epsilon \rightarrow 0$. Furthermore, due to the property (2.17), we have $X^\dagger = -X$ and therefore X takes values in \mathfrak{f} provided γ is real.

2.4 The deformed model

Non-split R -matrix. Equation (2.19) expresses X as a difference of the quantities $\mathcal{L}^g(\lambda_\pm)$ taking values in the Borel subalgebras \mathfrak{b}^\pm of $\mathfrak{f}^{\mathbb{C}}$. It turns out to be possible to invert this relation so as to express both $\mathcal{L}^g(\lambda_\pm)$ in terms of X by introducing a certain \mathbb{R} -linear operator on \mathfrak{f} .

To define this operator we begin by expressing the quantities $\mathcal{L}^g(\lambda_\pm)$ satisfying the properties (2.18) in terms of basis elements, namely

$$\mathcal{L}^g(\lambda_\pm) = \pm\gamma \left(\sum_{i=1}^n h_i H^i + \sum_{\alpha>0} e_{\pm\alpha} E^{\pm\alpha} \right). \tag{2.20}$$

We may then write X as defined by (2.19) more explicitly in terms of the basis (A.4) of \mathfrak{f} as

$$X = \sum_{i=1}^n h_i T^i + \frac{1}{2\sqrt{2}} \sum_{\alpha>0} \left((e_\alpha + e_{-\alpha}) B^\alpha + i(e_\alpha - e_{-\alpha}) C^\alpha \right). \tag{2.21}$$

Using the reality condition (A.3) we find $e_{-\alpha} = \bar{e}_\alpha$ so that all the above components of X in this basis are indeed real. If we now introduce an \mathbb{R} -linear operator $R : \mathfrak{f} \rightarrow \mathfrak{f}$ as follows [5]

$$R(T^i) = 0, \quad R(B^\alpha) = C^\alpha, \quad R(C^\alpha) = -B^\alpha, \tag{2.22}$$

then the sum of the quantities $\mathcal{L}^g(\lambda_\pm)$ is given simply by

$$RX = \frac{1}{2\gamma} (\mathcal{L}^g(\lambda_+) + \mathcal{L}^g(\lambda_-)). \tag{2.23}$$

The \mathbb{R} -linear map defined in (2.22) is an R -matrix of the so called ‘non-split’ type since it satisfies the following variant of the modified classical Yang-Baxter equation

$$[RM, RN] - R([RM, N] + [M, RN]) = [M, N]. \tag{2.24}$$

We refer to appendix C for a brief comparison of the properties of the R -matrix introduced here with the R -matrix of the ‘split’ type used, for instance, in [7]. Finally, combining equations (2.19) and (2.23) we may solve the pair of conditions (2.18) and write

$$\mathcal{L}^g(\lambda_{\pm}) = \gamma(R \mp i)X. \quad (2.25)$$

Lifting to (g, X) . It is now possible to explicitly relate the fields (g, X) introduced previously to the fields (j_0, j_1) used thus far. Doing so will, in particular, enable us to describe the Hamiltonian dynamics of (g, X) . Substituting the relation (2.25) into the expressions (2.15) for the gauge transformed Lax matrix at the points λ_{\pm} we obtain

$$\mathcal{L}(\lambda_{\pm}) = -g^{-1}\partial_{\sigma}g + \gamma g^{-1}((R \mp i)X)g. \quad (2.26)$$

On the other hand, the Lax matrix at these points can certainly be obtained directly in terms of the fields (j_0, j_1) since

$$\mathcal{L}(\lambda_{\pm}) = \frac{1}{1 - \lambda_{\pm}^2}(j_1 + \lambda_{\pm} j_0) = (1 - \epsilon^2)j_1 \pm i\epsilon\sqrt{1 - \epsilon^2}j_0.$$

Comparing the above two expressions for $\mathcal{L}(\lambda_{\pm})$ immediately yields the desired expressions for (j_0, j_1) in terms of (g, X) , namely

$$\begin{aligned} j_1 &= \frac{1}{1 - \epsilon^2}(-g^{-1}\partial_{\sigma}g + \gamma g^{-1}(RX)g), \\ j_0 &= -\frac{\gamma}{\epsilon\sqrt{1 - \epsilon^2}}g^{-1}Xg. \end{aligned}$$

If we fix $\gamma = -\epsilon(1 - \epsilon^2)^{3/2}$ then one can show that the full list of deformed Poisson brackets (2.10) for the components of the current (j_0, j_1) follows from the above relations and the following Poisson brackets for g and X ,

$$\{g_{\underline{1}}(\sigma), g_{\underline{2}}(\sigma')\}_{\epsilon} = 0, \quad (2.27a)$$

$$\{X_{\underline{1}}(\sigma), X_{\underline{2}}(\sigma')\}_{\epsilon} = [C_{\underline{1}\underline{2}}, X_{\underline{2}}(\sigma)]\delta_{\sigma\sigma'}, \quad (2.27b)$$

$$\{X_{\underline{1}}(\sigma), g_{\underline{2}}(\sigma')\}_{\epsilon} = C_{\underline{1}\underline{2}}g_{\underline{2}}(\sigma)\delta_{\sigma\sigma'}. \quad (2.27c)$$

To establish this result, one needs to use the fact that R is a non-split anti-symmetric R -matrix. This enables in particular to derive the following useful intermediate results,

$$\begin{aligned} \{(g^{-1}RXg)_{\underline{1}}(\sigma), (g^{-1}RXg)_{\underline{2}}(\sigma')\}_{\epsilon} &= [C_{\underline{1}\underline{2}}, (g^{-1}Xg)_{\underline{2}}]\delta_{\sigma\sigma'}, \\ \{(g^{-1}\partial_{\sigma}g)_{\underline{1}}(\sigma), (g^{-1}RXg)_{\underline{2}}(\sigma')\}_{\epsilon} &= -\{(g^{-1}RXg)_{\underline{1}}(\sigma), (g^{-1}\partial_{\sigma'}g)_{\underline{2}}(\sigma')\}_{\epsilon}. \end{aligned}$$

The final expressions for the components (j_0, j_1) in the deformed theory read

$$j_1 = -\frac{1}{1 - \epsilon^2}g^{-1}\partial_{\sigma}g - \epsilon\sqrt{1 - \epsilon^2}g^{-1}(RX)g, \quad (2.29a)$$

$$j_0 = (1 - \epsilon^2)g^{-1}Xg. \quad (2.29b)$$

We clearly see from these expressions that when $\epsilon \rightarrow 0$ we obtain the relation $j_1 = -g^{-1}\partial_\sigma g$ of the principal chiral model. On the other hand we also obtain $j_0 = g^{-1}Xg$ which identifies X with the component $l_0 = gj_0g^{-1}$ of the right invariant current in this limit. In particular, we see that the Poisson algebra (2.8) remains undeformed when $\epsilon \neq 0$ since (2.27) is exactly of the same form. Note by contrast that we no longer have $\{j_1(\sigma), g(\sigma')\}_\epsilon = 0$ when $\epsilon \neq 0$.

As previously anticipated, we explicitly observe the presence in (2.29) of a singularity at $\epsilon = 1$. In particular, if we insert the relations (2.29) into the Hamiltonian H_{PCM} of the principal chiral model we find that the resulting Hamiltonian of the deformed model is singular at $\epsilon = 1$.

Finally, the equations of motion for g and X are obtained by computing their Poisson brackets with the Hamiltonian H^ϵ . One finds

$$\partial_\tau gg^{-1} = -(1 - \epsilon^2)^2 \left(1 - \frac{\epsilon^2}{1 - \epsilon^2} R^2\right) X + \frac{\epsilon}{\sqrt{1 - \epsilon^2}} R(\partial_\sigma gg^{-1}), \quad (2.30a)$$

$$\partial_\tau X = \frac{1}{1 - \epsilon^2} \partial_\sigma (gj_1g^{-1}) - \epsilon \sqrt{1 - \epsilon^2} ([R(gj_1g^{-1}), X] + [gj_1g^{-1}, RX]). \quad (2.30b)$$

2.5 Symmetry algebra

Having completely defined the deformed model in the Hamiltonian formalism, we now turn to the description of its symmetries. In the principal chiral model, the generators of the global $F_L \times F_R$ symmetry can be conveniently extracted from the leading expansion of the monodromy at $\lambda = 0$ and $\lambda = \infty$, respectively. We will show that the symmetries of the deformed model with $\epsilon \neq 0$ can be similarly obtained by expanding the monodromy but at the points $\lambda = \lambda_\pm$ and $\lambda = \infty$.

Undeformed F_R symmetry. To begin with, consider the expansion of the Lax matrix at the point $\lambda = \infty$. At leading order it is given simply by (2.6), namely

$$\mathcal{L}(\lambda) = -\lambda^{-1}j_0 + O(\lambda^{-2}).$$

Thus the expansion of the monodromy at $\lambda = \infty$ will start with the same local charges $\int d\sigma j_0$ as in the undeformed theory. However, referring back to the deformed Poisson algebra (2.10a) we see that it is natural to scale these charges for $\epsilon \neq 0$ by defining

$$Q^R = \frac{1}{1 - \epsilon^2} \int d\sigma j_0. \quad (2.31)$$

The charges (2.31) so defined then satisfy the same Poisson algebra at all values of the deformation parameter ϵ . Moreover, these charges generate the same F_R symmetry on the group element g .

Deformed F_L symmetry: charges. Next, we consider how the F_L symmetry of the principal chiral model is affected by the deformation. We shall do this in two steps. We start by identifying the relevant conserved charges and subsequently proceed to determine their Poisson algebra.

A convenient way of extracting these charges in the principal chiral model is to first perform a gauge transformation by the principal chiral field and then read off the charges

from the expansion of the gauge transformed monodromy at $\lambda = 0$. As explained above, the double pole of the twist function at $\lambda = 0$ gets replaced in the deformed theory by the two single poles at $\lambda = \lambda_{\pm}$ of the deformed twist function. In light of all this, a natural prescription for extracting the corresponding charges in the deformed theory is to first perform a gauge transformation by the group valued field g and consider the expansion of the gauge transformed monodromy at the points $\lambda = \lambda_{\pm}$.

We shall therefore consider the expansions of the gauge transformed Lax matrix $\mathcal{L}^g(\lambda)$ around λ_{\pm} . The first thing to note is that since the leading terms of these expansions are non-zero, the extraction of the corresponding charges is far more involved. This is to be contrasted with the situation in the principal chiral model where the expansion of the gauge transformed Lax matrix at $\lambda = 0$ starts with $\mathcal{L}^g(\lambda) = -\lambda l_0 + O(\lambda^2)$. However, the important point is that although $\mathcal{L}^g(\lambda_{\pm})$ are both non-zero, they each live in a Borel subalgebra of $\mathfrak{f}^{\mathbb{C}}$. This will enable us to extract individual charges directly from the path ordered exponential entering the definition of the gauge transformed monodromy at these points.

Specifically, if $T(\lambda)$ is the monodromy, then the gauge transformed monodromy at λ_{\pm} reads

$$T^g(\lambda_{\pm}) = g(\infty)T(\lambda_{\pm})g(-\infty)^{-1} = P\overleftarrow{\text{exp}} \left[\int_{-\infty}^{\infty} d\sigma \mathcal{L}^g(\lambda_{\pm}) \right].$$

Recalling the expressions (2.20) for the gauge transformed Lax matrix, namely

$$\mathcal{L}^g(\lambda_{\pm}) = \pm\gamma \left(\sum_{i=1}^n h_i H^i + \sum_{\alpha>0} e_{\pm\alpha} E^{\pm\alpha} \right),$$

we will show that the Cartan components of $\mathcal{L}^g(\lambda_{\pm})$ can be factored out of the above path ordered exponential. For this, we will use the following identity, valid for any functions ϕ_i and $L_{\pm\alpha}$ of σ ,

$$P\overleftarrow{\text{exp}} \left[\int_{\sigma_1}^{\sigma_2} d\sigma \left(\sum_{i=1}^n (\partial_{\sigma} \phi_i) H^i + \sum_{\alpha>0} L_{\pm\alpha} E^{\pm\alpha} \right) \right] = \exp \left(\sum_{i=1}^n \phi_i(\sigma_2) H^i \right) \times P\overleftarrow{\text{exp}} \left[\int_{\sigma_1}^{\sigma_2} d\sigma \sum_{\alpha>0} e^{\mp \sum_{i=1}^n \alpha(H^i)\phi_i(\sigma)} L_{\pm\alpha} E^{\pm\alpha} \right] \exp \left(- \sum_{i=1}^n \phi_i(\sigma_1) H^i \right). \quad (2.32)$$

To apply this identity to the path ordered exponential of $\mathcal{L}^g(\lambda_{+})$ we let $\phi_i(\sigma) = \int_{-\infty}^{\sigma} d\sigma' \gamma h_i(\sigma')$ and $L_{\alpha}(\sigma) = \gamma e_{\alpha}(\sigma)$. Then taking $\sigma_1 = -\infty$ and $\sigma_2 = \infty$ in (2.32) gives

$$T^g(\lambda_{+}) = \exp \left(\gamma \int_{-\infty}^{\infty} d\sigma \sum_{i=1}^n h_i(\sigma) H^i \right) P\overleftarrow{\text{exp}} \left[\gamma \sum_{\alpha>0} \int_{-\infty}^{\infty} d\sigma \mathfrak{J}_{\alpha}^E(\sigma) E^{\alpha} \right], \quad (2.33a)$$

where the quantity $\mathfrak{J}_{\alpha}^E(\sigma)$ is defined below. Similarly, to describe the path ordered exponential of $\mathcal{L}^g(\lambda_{-})$ we choose $\phi_i(\sigma) = \int_{\sigma}^{\infty} d\sigma' \gamma h_i(\sigma')$ and $L_{-\alpha}(\sigma) = -\gamma e_{-\alpha}(\sigma)$. Letting $\sigma_1 = -\infty$ and $\sigma_2 = \infty$ in (2.32) we obtain

$$T^g(\lambda_{-}) = P\overleftarrow{\text{exp}} \left[-\gamma \sum_{\alpha>0} \int_{-\infty}^{\infty} d\sigma \mathfrak{J}_{-\alpha}^E(\sigma) E^{-\alpha} \right] \exp \left(-\gamma \int_{-\infty}^{\infty} d\sigma \sum_{i=1}^n h_i(\sigma) H^i \right), \quad (2.33b)$$

where the notation is as follows. For any positive root $\alpha > 0$ we let

$$\mathfrak{J}_\alpha^H(\sigma) = \sum_{i=1}^n \alpha(H^i) h_i(\sigma), \quad \mathfrak{J}_{\pm\alpha}^E(\sigma) = e_{\pm\alpha}(\sigma) e^{-\gamma\chi_\alpha(\sigma)} e^{\gamma\chi_\alpha(\mp\infty)}. \quad (2.34)$$

The function χ_α has the property that $\partial_\sigma \chi_\alpha(\sigma) = \mathfrak{J}_\alpha^H(\sigma)$ and is defined explicitly by

$$\chi_\alpha(\sigma) = \frac{1}{2} \int_{-\infty}^{\infty} d\sigma' \epsilon_{\sigma\sigma'} \mathfrak{J}_\alpha^H(\sigma') = \int_{-\infty}^{\sigma} d\sigma' \mathfrak{J}_\alpha^H(\sigma') - \frac{1}{2} \int_{-\infty}^{\infty} d\sigma' \mathfrak{J}_\alpha^H(\sigma').$$

Here we use the notation $\epsilon_{\sigma\sigma'} = \epsilon(\sigma - \sigma')$ which satisfies $\partial_\sigma \epsilon_{\sigma\sigma'} = 2\delta_{\sigma\sigma'}$. The boundary values of the function χ_α at $\pm\infty$ are

$$\chi_\alpha(\pm\infty) = \pm \frac{1}{2} \int_{-\infty}^{\infty} d\sigma' \mathfrak{J}_\alpha^H(\sigma'). \quad (2.35)$$

Note that the transformation $h_i(\sigma) \rightarrow \mathfrak{J}_{\alpha_i}^H(\sigma)$ is invertible since the symmetrized Cartan matrix B_{ij} is invertible, namely we can write $h_i(\sigma) = \sum_{j=1}^n B_{ij}^{-1} \mathfrak{J}_{\alpha_j}^H(\sigma)$ (see appendix A for notations).

The advantage of the factorized form (2.33) is that the argument in the remaining path ordered exponential on the right hand side is nilpotent. Therefore, this path ordered exponential can now be evaluated explicitly in terms of exponentials of ordinary integrals. In particular, this allows one to define charges Q_α^E corresponding to each root $\alpha \in \Phi$, the conservation of which then follows from the conservation of $T^g(\lambda_\pm)$. Details of the procedure for defining these charges can be found in appendix B. In the remainder of this section, however, we will only be needing the conserved charges associated with the Cartan generators and the simple roots. These are given by

$$\int_{-\infty}^{\infty} d\sigma \mathfrak{J}_{\alpha_i}^H(\sigma) \quad \text{and} \quad \int_{-\infty}^{\infty} d\sigma \mathfrak{J}_{\pm\alpha_i}^E(\sigma)$$

where the α_i , $i = 1, \dots, n$ are the simple roots of $\mathfrak{f}^{\mathbb{C}}$.

Let us remark that the conservation of the quantities $\int_{-\infty}^{\infty} d\sigma h_i(\sigma)$ could also be shown relatively straightforwardly from their definitions. Indeed, one can check that the projection of both sides of the equation of motion (2.30b) for X onto \mathfrak{h} together with (2.21) and (2.22) lead to the desired conservation property.

Deformed F_L symmetry: algebra. In the remainder of this section we determine the Poisson algebra of the charges identified above.

The Lax matrix $\mathcal{L}^g(\lambda_\pm)$ as given in (2.25) only depends on the field X , whose expression (2.21) can be rewritten as

$$X = \sum_{j=1}^n i h_j H^j + \frac{i}{2} \sum_{\alpha>0} (e_\alpha E^{+\alpha} + e_{-\alpha} E^{-\alpha}). \quad (2.36)$$

It is apparent from this that the Poisson bracket relations of the corresponding charges will follow solely from the Kostant-Kirillov Poisson bracket (2.27b). Using equation (A.1), this

Poisson bracket takes the following form

$$\{X_{\underline{1}}(\sigma), X_{\underline{2}}(\sigma')\}_\epsilon = \left(\sum_{i,j=1}^n B_{ij}^{-1} H^i \otimes [H^j, X(\sigma)] + \sum_{\alpha>0} (E^\alpha \otimes [E^{-\alpha}, X(\sigma)] + E^{-\alpha} \otimes [E^\alpha, X(\sigma)]) \right) \delta_{\sigma\sigma'}.$$

Comparing coefficients on both sides for the different basis elements of $\mathfrak{f}^{\mathbb{C}}$ in the first tensor factor then gives

$$\{h_i(\sigma), X(\sigma')\}_\epsilon = -i \sum_{j=1}^n B_{ij}^{-1} [H^j, X(\sigma)] \delta_{\sigma\sigma'}, \quad \{e_{\pm\alpha}(\sigma), X(\sigma')\}_\epsilon = -2i [E^{\mp\alpha}, X(\sigma)] \delta_{\sigma\sigma'}. \quad (2.37)$$

Consider the first of these two relations. Using again (2.36) the comparison of the coefficients of H^j and $E^{\pm\alpha_j}$ on both sides leads respectively to

$$\{h_i(\sigma), h_j(\sigma')\}_\epsilon = 0, \quad \{h_i(\sigma), e_{\pm\alpha_j}(\sigma')\}_\epsilon = \mp i e_{\pm\alpha_j}(\sigma) \delta_{ij} \delta_{\sigma\sigma'}.$$

The second of these relations then implies

$$\{e^{-\gamma\chi_\alpha(\sigma)}, e_{\pm\alpha_j}(\sigma')\}_\epsilon = \pm \frac{i}{2} \gamma e_{\pm\alpha_j}(\sigma') \alpha(H^j) e^{-\gamma\chi_\alpha(\sigma)} \epsilon_{\sigma\sigma'}.$$

Likewise, specialising the second relation in (2.37) to the simple root α_i and comparing coefficients of $E^{-\alpha_j}$ on both sides gives

$$\{e_{\alpha_i}(\sigma), e_{-\alpha_j}(\sigma')\}_\epsilon = -4i \partial_\sigma \chi_{\alpha_i}(\sigma) \delta_{ij} \delta_{\sigma\sigma'}.$$

This allows us to compute commutation relations between the charge densities $\mathfrak{J}_{\pm\alpha_i}^E(\sigma)$ and $\mathfrak{J}_{\alpha_i}^H(\sigma)$, yielding

$$\begin{aligned} \{\mathfrak{J}_{\alpha_i}^E(\sigma), \mathfrak{J}_{-\alpha_j}^E(\sigma')\}_\epsilon &= -4i \partial_\sigma \chi_{\alpha_i}(\sigma) e^{-2\gamma\chi_{\alpha_i}(\sigma)} \delta_{ij} \delta_{\sigma\sigma'} \\ &= 2i \gamma^{-1} \partial_\sigma \left(e^{-2\gamma\chi_{\alpha_i}(\sigma)} \right) \delta_{ij} \delta_{\sigma\sigma'}, \end{aligned} \quad (2.38a)$$

$$\{\mathfrak{J}_{\alpha_i}^H(\sigma), \mathfrak{J}_{\pm\alpha_j}^E(\sigma')\}_\epsilon = \mp i B_{ij} \mathfrak{J}_{\pm\alpha_j}^E(\sigma') \delta_{\sigma\sigma'}. \quad (2.38b)$$

We now define the integrated charges from the above densities, namely

$$Q_{\alpha_i}^H = d_i^{-1} \int_{-\infty}^{\infty} d\sigma \mathfrak{J}_{\alpha_i}^H(\sigma), \quad Q_{\pm\alpha_i}^E = D_i \int_{-\infty}^{\infty} d\sigma \mathfrak{J}_{\pm\alpha_i}^E(\sigma), \quad (2.39)$$

where we define the notational shorthand

$$D_i = \left(\frac{\gamma}{4 \sinh(d_i \gamma)} \right)^{\frac{1}{2}}. \quad (2.40)$$

These normalisations in the charges $Q_{\pm\alpha_i}^E$ have been introduced for convenience (c.f. [2]). The Poisson brackets (2.38) for the densities then lead to

$$i\{Q_{\alpha_i}^H, Q_{\alpha_j}^H\}_\epsilon = 0, \tag{2.41a}$$

$$i\{Q_{+\alpha_i}^E, Q_{-\alpha_j}^E\}_\epsilon = \delta_{ij} \frac{q^{d_i Q_{\alpha_i}^H} - q^{-d_i Q_{\alpha_i}^H}}{q^{d_i} - q^{-d_i}}, \tag{2.41b}$$

$$i\{Q_{\alpha_i}^H, Q_{\pm\alpha_j}^E\}_\epsilon = \pm A_{ij} Q_{\pm\alpha_j}^E. \tag{2.41c}$$

Here we have made use of the values (2.35) and introduced the new parameter

$$q = e^\gamma = \exp\left(-\epsilon(1 - \epsilon^2)^{\frac{3}{2}}\right).$$

Furthermore, the charges $Q_{\alpha_i}^E$ also satisfy certain q -Poisson-Serre relations. To write these down we introduce a q -analogue of the deformed Poisson bracket as follows. We say that $Q_{\alpha_i}^E$ defined in (2.39) is associated with the simple root α_i . Let A_α and A_β denote charges associated with any pair of positive roots $\alpha, \beta > 0$ and define their q -Poisson bracket as

$$(\text{ad}_{\{\cdot, \cdot\}_{q^\epsilon}} A_\alpha)(A_\beta) := \{A_\alpha, A_\beta\}_{q^\epsilon} := \{A_\alpha, A_\beta\}_\epsilon + i\gamma(\alpha, \beta)A_\alpha A_\beta. \tag{2.42}$$

If $\alpha + \beta$ is a root then we regard the resulting quantity $\{A_\alpha, A_\beta\}_{q^\epsilon}$ as being associated with this root. The operator $(\text{ad}_{\{\cdot, \cdot\}_{q^\epsilon}} A_\alpha)^n$ may then be defined recursively for any $n \geq 1$. Using this notation, the q -Poisson-Serre relations can be written succinctly as follows

$$(\text{ad}_{\{\cdot, \cdot\}_{q^\epsilon}} Q_{\alpha_i}^E)^{1-A_{ij}}(Q_{\alpha_j}^E) = 0. \tag{2.43}$$

This identity is proved for all classical Lie algebras \mathfrak{f} in appendix B.

Finally, the charges (2.39) have the following behaviour under complex conjugation

$$\overline{Q_{\alpha_i}^H} = Q_{\alpha_i}^H, \quad \overline{Q_{\alpha_i}^E} = q^{-d_i Q_{\alpha_i}^H} Q_{-\alpha_i}^E, \tag{2.44}$$

which is easily seen to preserve the relations (2.41). The q -Poisson-Serre relations (2.43) are also mapped to the corresponding relations for negative roots. These take the form

$$(\text{ad}_{\{\cdot, \cdot\}_{q^{-1}\epsilon}} Q_{-\alpha_i}^E)^{1-A_{ij}}(Q_{-\alpha_j}^E) = 0,$$

where similarly to (2.42) we define the q -Poisson bracket of any two charges $A_{-\alpha}$ and $A_{-\beta}$ associated with the negative roots $-\alpha, -\beta < 0$ as

$$(\text{ad}_{\{\cdot, \cdot\}_{q^{-1}\epsilon}} A_{-\alpha})(A_{-\beta}) := \{A_{-\alpha}, A_{-\beta}\}_{q^{-1}\epsilon} := \{A_{-\alpha}, A_{-\beta}\}_\epsilon - i\gamma(\alpha, \beta)A_{-\alpha}A_{-\beta}.$$

Interpretation as semiclassical limit of $U_{\hat{q}}(\mathfrak{f})$. The algebra of the deformed F_L symmetry just obtained bears a strikingly resemblance with the relations of the quantum group $U_q(\mathfrak{f})$, but where the commutators are replaced by Poisson brackets. To close the discussion on symmetries, we will show that the Poisson algebra generated by the charges $Q_{\alpha_i}^H$ and $Q_{\pm\alpha_i}^E$, subject to the relations (2.41), (2.43) and (2.44), coincides exactly with the semiclassical limit $\hbar \rightarrow 0$ of the compact real form $U_{\hat{q}}(\mathfrak{f})$ of the quantum group $U_{\hat{q}}(\mathfrak{f}^{\mathbb{C}})$ where

$\hat{q} = q^{\hbar}$. The resulting Poisson algebra, which we shall denote $U_q^P(\mathfrak{f})$, gives a one-parameter deformation of the Poisson algebra corresponding to the Lie algebra \mathfrak{f} . Similar semiclassical limits of finite dimensional quantum groups were considered in [11–13] and the case of the quantum affine algebra $U_{\hat{q}}(\widehat{\mathfrak{sl}}_2)$ in [14].

Recall that $U_{\hat{q}}(\mathfrak{f}^{\mathbb{C}})$ is generated by $\widehat{H}_i, \widehat{E}_i, \widehat{F}_i$ for $i = 1, \dots, n = \text{rk } \mathfrak{f}^{\mathbb{C}}$ subject to the relations

$$[\widehat{E}_i, \widehat{F}_j] = \delta_{ij} \frac{\widehat{K}_i - \widehat{K}_i^{-1}}{\widehat{q}^{d_i} - \widehat{q}^{-d_i}}, \quad [\widehat{H}_i, \widehat{H}_j] = 0, \quad (2.45a)$$

$$[\widehat{H}_i, \widehat{E}_j] = A_{ij} \widehat{E}_j, \quad [\widehat{H}_i, \widehat{F}_j] = -A_{ij} \widehat{F}_j, \quad (2.45b)$$

where $\widehat{K}_i = \widehat{q}^{d_i \widehat{H}_i}$, together with the \widehat{q} -Serre relations which may be written as [15]

$$(\text{ad}_{[\cdot, \cdot]_{\widehat{q}}} \widehat{E}_i)^{1-A_{ij}}(\widehat{E}_j) = 0, \quad (\text{ad}_{[\cdot, \cdot]_{\widehat{q}^{-1}}} \widehat{F}_i)^{1-A_{ij}}(\widehat{F}_j) = 0. \quad (2.46)$$

Here we have introduced the \widehat{q} -analog of the commutator along with the corresponding \widehat{q} -analog of the adjoint action as

$$(\text{ad}_{[\cdot, \cdot]_{\widehat{q}^{\pm 1}}} \widehat{A}_\alpha)(\widehat{A}_\beta) := [\widehat{A}_\alpha, \widehat{A}_\beta]_{\widehat{q}^{\pm 1}} := \widehat{A}_\alpha \widehat{A}_\beta - \widehat{q}^{\pm(\alpha, \beta)} \widehat{A}_\beta \widehat{A}_\alpha, \quad (2.47)$$

where the $+$ (respectively $-$) sign is used if the roots α, β are positive (respectively negative).

There are many possible Hopf algebra structures on $U_q(\mathfrak{f}^{\mathbb{C}})$ corresponding to different choices of coproducts. The real structures on $U_q(\mathfrak{f}^{\mathbb{C}})$ have been classified in [16] with respect to the standard coproduct [17], but other choices of coproducts lead to alternative reality conditions [18]. For our purposes we shall consider the coproduct defined on the generators as [15, 19]

$$\Delta \widehat{E}_i = \widehat{E}_i \otimes 1 + \widehat{K}_i^{-1} \otimes \widehat{E}_i, \quad \Delta \widehat{F}_i = \widehat{F}_i \otimes \widehat{K}_i + 1 \otimes \widehat{F}_i, \quad \Delta \widehat{H}_i = \widehat{H}_i \otimes 1 + 1 \otimes \widehat{H}_i. \quad (2.48)$$

The compact real form $U_{\widehat{q}}(\mathfrak{f})$ of $U_{\widehat{q}}(\mathfrak{f}^{\mathbb{C}})$ then corresponds to the condition $\widehat{q} \in \mathbb{R}$ and the following choice of $*$ -involution on $U_{\widehat{q}}(\mathfrak{f}^{\mathbb{C}})$ [19]

$$\widehat{K}_i^* = \widehat{K}_i, \quad \widehat{E}_i^* = \widehat{K}_i^{-1} \widehat{F}_i, \quad \widehat{F}_i^* = \widehat{E}_i \widehat{K}_i. \quad (2.49)$$

To take the semiclassical limit $\hbar \rightarrow 0$ of the above relations we suppose the generators $\widehat{H}_i, \widehat{E}_i$ and \widehat{F}_i have the following leading order behaviour in this limit

$$\hbar \widehat{H}_i \longrightarrow Q_{\alpha_i}^H, \quad \hbar \widehat{E}_i \longrightarrow \left(\frac{\sinh(d_i \gamma)}{d_i \gamma} \right)^{\frac{1}{2}} Q_{\alpha_i}^E, \quad \hbar \widehat{F}_i \longrightarrow \left(\frac{\sinh(d_i \gamma)}{d_i \gamma} \right)^{\frac{1}{2}} Q_{-\alpha_i}^E.$$

Moreover, we also assume the leading behaviour of the commutator to correspond to the deformed Poisson bracket (2.9), namely

$$\frac{1}{\hbar} [\cdot, \cdot] \longrightarrow i \{ \cdot, \cdot \}_\epsilon.$$

It is now easy to see that the relations (2.45b) reproduce (2.41c) in the limit $\hbar \rightarrow 0$. Furthermore, owing to the normalisations of the generators \widehat{E}_i and \widehat{F}_i in this limit we recover also (2.41b) from the first relation in (2.45a).

Using the relation $\widehat{q} = q^\hbar$, we find that the leading behaviour of the \widehat{q} -commutator (2.47) is given by the q -Poisson bracket (2.42), that is

$$\frac{1}{\hbar}[\cdot, \cdot]_{\widehat{q}^{\pm 1}} \longrightarrow i\{\cdot, \cdot\}_{q^{\pm 1} \epsilon}.$$

It directly follows from this that the semiclassical limit of the \widehat{q} -Serre relations (2.46) is exactly the q -Poisson-Serre relations (2.43). The reality conditions (2.49) also lead to (2.44) in this limit.

Finally, taking the semiclassical limit of the coproduct (2.48) we obtain

$$\Delta Q_{\alpha_i}^E = Q_{\alpha_i}^E \otimes 1 + q^{-d_i Q_{\alpha_i}^H} \otimes Q_{\alpha_i}^E, \quad (2.50a)$$

$$\Delta Q_{-\alpha_i}^E = Q_{-\alpha_i}^E \otimes q^{d_i Q_{\alpha_i}^H} + 1 \otimes Q_{-\alpha_i}^E, \quad (2.50b)$$

$$\Delta Q_{\alpha_i}^H = Q_{\alpha_i}^H \otimes 1 + 1 \otimes Q_{\alpha_i}^H. \quad (2.50c)$$

Equipped with this choice of coproduct, the real Poisson algebra $U_q^{\mathcal{P}}(\mathfrak{f})$, defined by the relations (2.41), (2.43) and the real structure (2.44) on the charges $Q_{\alpha_i}^H$ and $Q_{\pm\alpha_i}^E$, acquires the structure of a real Poisson-Hopf algebra.

2.6 Yang-Baxter σ -model

In this subsection, we will show that the deformed model coincides with the Yang-Baxter σ -model introduced by Klimčík in [4, 5]. For this we need to perform the inverse Legendre transform from the Hamiltonian formalism to the Lagrangian formalism.

Lagrangian. The inverse Legendre transform is given by

$$L = \kappa(\partial_\tau g g^{-1}, X) - h^\epsilon \quad (2.51)$$

where the Hamiltonian density $h^\epsilon = h^{PCM}$, defined by equation (2.2), can be re-expressed in terms of the light-cone components $j_\pm = j_0 \pm j_1$ of the current as

$$h^\epsilon = -\frac{1}{4}\kappa(j_+, j_+) - \frac{1}{4}\kappa(j_-, j_-). \quad (2.52)$$

As usual, to rewrite (2.51) in terms of Lagrangian fields we begin by expressing X in terms of g and its time derivative. This can be done using the equation of motion (2.30a). It turns out to be convenient to express everything in terms of the following variable

$$\eta = \frac{\epsilon}{\sqrt{1 - \epsilon^2}}. \quad (2.53)$$

Noting that $1 \pm \eta R$ is invertible since R is a real skew-symmetric operator and therefore has only imaginary eigenvalues, one obtains

$$X = -\frac{1}{2}(1 + \eta^2)^2 \left(\frac{1}{1 - \eta R} \partial_- g g^{-1} + \frac{1}{1 + \eta R} \partial_+ g g^{-1} \right) \quad (2.54)$$

with $\partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}$. Using this result we may also express j_{\pm} in terms of Lagrangian fields. Starting from equations (2.29) we have

$$j_{\pm} = \mp \frac{1}{1 - \epsilon^2} g^{-1} \partial_{\sigma} g + (1 - \epsilon^2) g^{-1} (1 \mp \eta R) X g. \quad (2.55)$$

Then combining equations (2.55) and (2.54) we find

$$g j_{\pm} g^{-1} = -\frac{1}{1 - \epsilon^2} \frac{1}{1 \pm \eta R} \partial_{\pm} g g^{-1}. \quad (2.56)$$

The last step consists in substituting (2.54) and (2.56) into the expression (2.51) for the Lagrangian. This yields the Lagrangian of the deformed model,

$$L = -\frac{1}{2} \kappa \left(\partial_{+} g g^{-1}, \frac{(1 + \eta^2)^2}{1 - \eta R} \partial_{-} g g^{-1} \right) \quad (2.57)$$

where the operator R is the non-split R -matrix defined by equation (2.22) and η is expressed in terms of the deformation parameter ϵ as (2.53). This corresponds to the Yang-Baxter σ -model defined by Klimčík in [4, 5]. Finally, note that when ϵ tends to zero the Lagrangian (2.57) reduces to that of the principal chiral model.

Comments. To close our discussion on the deformation of the principal chiral model, we compare our definition of the field g given in section 2.3 with the corresponding definition in [5].

Consider the extended solution $\Psi(\lambda, \sigma)$ of the principal chiral model, which by definition solves the auxiliary linear problem

$$\partial_{\sigma} \Psi(\lambda) \Psi(\lambda)^{-1} = \mathcal{L}(\lambda), \quad \Psi(\lambda, 0) = \mathbf{1}.$$

Since the Lax matrix (2.4) has the property that $\mathcal{L}(0) = j_1 = -g^{-1} \partial_{\sigma} g$, it follows that the principal chiral field g (or rather its inverse) can be recovered from the extended solution evaluated at $\lambda = 0$, namely $g^{-1} = \Psi(0)$. Similarly, it was shown in [5] that the field g of the Yang-Baxter σ -model can also be retrieved from the same extended solution $\Psi(\lambda)$ of the principal chiral model, but evaluated instead at the special point $\lambda = i\eta$. More precisely, g^{-1} coincides with the element of the compact subgroup F in the Iwasawa decomposition of $\Psi(i\eta) \in F^{\mathbb{C}}$. To see why this definition agrees with ours, note first that $i\eta$ corresponds to the pole λ_{+} . Letting $\Psi(i\eta) = g^{-1} a n$ be the Iwasawa decomposition, where g and $a n$ respectively take values in F and the Borel subgroup $B^{+} = \exp \mathfrak{b}^{+} \subset F^{\mathbb{C}}$, we may then write

$$\mathcal{L}(i\eta) = \partial_{\sigma} \Psi(i\eta) \Psi(i\eta)^{-1} = \partial_{\sigma} (g^{-1}) g + g^{-1} (\partial_{\sigma} (a n) (a n)^{-1}) g.$$

But this agrees precisely with (2.26) which can be rewritten as

$$\mathcal{L}(i\eta) = \partial_{\sigma} (g^{-1}) g + g^{-1} (\gamma(R - i) X) g.$$

In particular, we have the identification $\partial_{\sigma} (a n) (a n)^{-1} = \gamma(R - i) X$ as elements in \mathfrak{b}^{+} .

It is now apparent that one of the virtues of our approach lies in the identification of the special points $\pm i\eta$ with the poles of the twist function. This will be fully exploited in the next section to extend the above analysis and construct an integrable deformation of coset σ -models.

3 Deforming symmetric space σ -models

In this section we discuss the deformation of symmetric space σ -models, following a very similar approach to the one developed in the previous section for deforming the principal chiral model. For this reason, we insist more on the new aspects related to the case at hand and omit details which are similar to the previous case. We use the conventions and notations of [7].

3.1 Symmetric space σ -models

Hamiltonian and Lax matrix. Let F be a compact Lie group with Lie algebra \mathfrak{f} . We equip \mathfrak{f} with a \mathbb{Z}_2 -automorphism σ so that $\sigma^2 = \text{id}$. This induces the usual decomposition $\mathfrak{f} = \mathfrak{f}^{(0)} \oplus \mathfrak{f}^{(1)}$ into the eigenspaces of σ where $\mathfrak{f}^{(0)} = \mathfrak{g}$ is a Lie subalgebra with corresponding Lie group $G = \exp \mathfrak{g}$. Let P_0 and P_1 denote the projection operators onto the respective subspaces $\mathfrak{f}^{(0)}$ and $\mathfrak{f}^{(1)}$ relative to this decomposition.

We consider the coset σ -model on the symmetric space F/G . It is described by a pair of fields A and Π valued in \mathfrak{f} . The Poisson structure on the graded components $A^{(0)}, A^{(1)}$ and $\Pi^{(0)}, \Pi^{(1)}$ of these fields reads

$$\{A_{\underline{1}}^{(i)}(\sigma), A_{\underline{2}}^{(j)}(\sigma')\} = 0, \tag{3.1a}$$

$$\{A_{\underline{1}}^{(i)}(\sigma), \Pi_{\underline{2}}^{(j)}(\sigma')\} = [C_{\underline{1}\underline{2}}^{(ii)}, A_{\underline{2}}^{(i+j)}(\sigma)] \delta_{\sigma\sigma'} - C_{\underline{1}\underline{2}}^{(ii)} \delta_{ij} \delta'_{\sigma\sigma'}, \tag{3.1b}$$

$$\{\Pi_{\underline{1}}^{(i)}(\sigma), \Pi_{\underline{2}}^{(j)}(\sigma')\} = [C_{\underline{1}\underline{2}}^{(ii)}, \Pi_{\underline{2}}^{(i+j)}(\sigma)] \delta_{\sigma\sigma'}. \tag{3.1c}$$

Here $C_{\underline{1}\underline{2}}^{(ii)}$ are the graded components of the Casimir (A.1) with respect to the automorphism σ .

The Hamiltonian of the coset σ -model is

$$H_{\text{coset}} = \int_{-\infty}^{\infty} d\sigma \left[T_{++} + T_{--} + \kappa(A^{(0)}, \Pi^{(0)}) + \kappa(\ell, \Pi^{(0)}) \right] \tag{3.2}$$

where $T_{\pm\pm} = -\frac{1}{4}\kappa(A_{\pm}^{(1)}, A_{\pm}^{(1)})$ and $A_{\pm}^{(1)} = \Pi^{(1)} \mp A^{(1)}$. The field ℓ is a Lagrange multiplier associated with the constraint $\Pi^{(0)}$ corresponding to the coset gauge invariance.

The integrability of this model is encoded at the Hamiltonian level in the Lax matrix [20]

$$\mathcal{L}(\lambda) = A^{(0)} + \frac{1}{2}(\lambda^{-1} + \lambda)A^{(1)} + \frac{1}{2}(1 - \lambda^2)\Pi^{(0)} + \frac{1}{2}(\lambda^{-1} - \lambda)\Pi^{(1)}. \tag{3.3}$$

It has the following property with respect to the automorphism

$$\mathcal{L}(-\lambda) = \sigma(\mathcal{L}(\lambda)), \tag{3.4}$$

which amounts to saying that $\mathcal{L}(\lambda)$ takes values in the twisted loop algebra $\widehat{\mathfrak{f}}^{\sigma}$.

Symmetry algebra. The coset σ -model on F/G is invariant under a global F_L symmetry. The corresponding conserved charges may be extracted from the leading behaviour of the monodromy matrix at the point $\lambda = 1$. Indeed, the expansion of the Lax matrix at this point reads

$$\mathcal{L}(\lambda) = A - (\lambda - 1)\Pi + O((\lambda - 1)^2). \tag{3.5}$$

If we introduce the group valued field g through the relation $A = -g^{-1}\partial_\sigma g$, and on which the F_L symmetry acts as $g \mapsto U_L g$, then the gauged transformed Lax matrix $\mathcal{L}^g(\lambda)$ previously defined in (2.7) satisfies

$$\mathcal{L}^g(\lambda) = (\lambda - 1)X + O((\lambda - 1)^2) \tag{3.6}$$

with $X = -g\Pi g^{-1}$. The coefficient of $\lambda - 1$ in the expansion of the gauge transformed monodromy matrix then yields the generator of the F_L symmetry, namely $Q^L = \int d\sigma X$.

3.2 Setting up the deformation

Poisson bracket. Following the same strategy as for the principal chiral model, we will deform the Poisson bracket $\{\cdot, \cdot\}$ of (A, Π) in (3.1) by adding to it the generalized Faddeev-Reshetikhin Poisson bracket $\{\cdot, \cdot\}'$ introduced recently in [7]. Since these two brackets are compatible, any linear combination still defines a Poisson bracket. We therefore set

$$\{\cdot, \cdot\}_\epsilon := \{\cdot, \cdot\} + \epsilon^2 \{\cdot, \cdot\}' \tag{3.7}$$

where ϵ is a positive real deformation parameter. The explicit form of this Poisson bracket on the fields A and Π is given in appendix D.

Lax matrix and Hamiltonian. We shall suppose, as we did in the principal chiral model case, that the dependence of the Lax matrix $\mathcal{L}(\lambda)$ on the fields (A, Π) does not change with ϵ . Moreover, we also impose that the dynamics of the fields (A, Π) remains the same as we turn on the deformation. These two requirements ensure that the dynamics of the deformed model remains integrable for $\epsilon \neq 0$.

Therefore, the Hamiltonian H^ϵ , which generates the same dynamics on the fields (A, Π) as the coset σ -model but with respect to the interpolating bracket $\{\cdot, \cdot\}_\epsilon$, should satisfy

$$\{H^\epsilon, \mathcal{L}\}_\epsilon = \{H_{\text{coset}}, \mathcal{L}\}. \tag{3.8}$$

Postulating a general quadratic ansatz for H^ϵ we find the unique solution of this equation to be

$$H^\epsilon = H_{\text{coset}} + \epsilon^2 \int d\sigma \kappa(\Pi^{(0)}, \Pi^{(0)}). \tag{3.9}$$

Plugging (3.9) directly into (3.8) and using the fact that $\Pi^{(0)}$ has a vanishing generalised Faddeev-Reshetikhin Poisson bracket with every function of (A, Π) , we see that the property (3.8) boils down to the following relation

$$\{H_{\text{coset}}, \mathcal{L}\}' = \left\{ - \int d\sigma \kappa(\Pi^{(0)}, \Pi^{(0)}), \mathcal{L} \right\},$$

which can be checked directly. Note that the Hamiltonian (3.9) satisfies the equation (3.8) strongly. That is to say, the equations of motion generated by the original Hamiltonian H_{coset} with respect to the original Poisson bracket $\{\cdot, \cdot\}$ are reproduced exactly, including terms proportional to the Hamiltonian constraint $\Pi^{(0)}$.

3.3 Defining the group valued field

So far we have merely discussed the dynamics of the coset σ -model with respect to the deformed Poisson bracket at the level of the fields (A, Π) . Following our procedure in the case of the principal chiral model, we anticipate the group valued field g in the deformed theory to correspond to the parameter of a gauge transformation of some sort. In fact, it is clear from the discussion leading to equation (3.6) that the field g of the coset σ -model may be described as the parameter of a gauge transformation sending the Lax matrix $\mathcal{L}(1)$ at $\lambda = 1$ to zero. To see how such a definition may be deformed when $\epsilon \neq 0$, we turn to the study of the twist function.

Deformed twist function and its poles. As before, the twist function of the deformed Poisson bracket can be expressed in terms of the twist functions of the two compatible Poisson brackets, namely [7]

$$\varphi_\sigma(\lambda) = \frac{4\lambda}{(1 - \lambda^2)^2}, \quad \varphi_{\text{gFR}}(\lambda) = \frac{1}{\lambda}. \tag{3.10}$$

The twist of the deformed model is then defined through the relation

$$\varphi_\epsilon^{-1} = \varphi_\sigma^{-1} + \epsilon^2 \varphi_{\text{gFR}}^{-1}.$$

Substituting the definitions (3.10) into this relation we find this twist function to be

$$\varphi_\epsilon(\lambda) = \frac{4\lambda}{\lambda^4 + (4\epsilon^2 - 2)\lambda^2 + 1}.$$

As we learned from the principal chiral model case, the poles of φ_ϵ will play an important role in defining the field g in the deformed theory as well as in extracting symmetry generators of the latter. In the present case we find a bifurcation in the behaviour of these poles at the special value $\epsilon = 1$. Specifically, for $0 \leq \epsilon < 1$, if we define an angle $0 \leq \theta < \frac{\pi}{2}$ by letting

$$\sin \theta = \epsilon$$

then the four poles λ_\pm and λ_\pm^{-1} of the twist function φ_ϵ are located on the unit circle, with

$$\lambda_\pm := \pm e^{i\theta}. \tag{3.11}$$

The two initial double poles of φ_0 at $\lambda = \pm 1$ (i.e. $\theta = 0$) therefore split into four distinct simple poles of φ_ϵ as we turn on the deformation parameter ϵ in the range $0 < \epsilon < 1$. But as ϵ approaches the value 1 (i.e. $\theta = \frac{\pi}{2}$), the four poles degenerate once again into two points at $\lambda = \pm i$. The behaviour of these poles is depicted in figure 1. As we increase ϵ further, for $\epsilon > 1$ we find that these double poles split once more into single poles and move off along the imaginary axis.

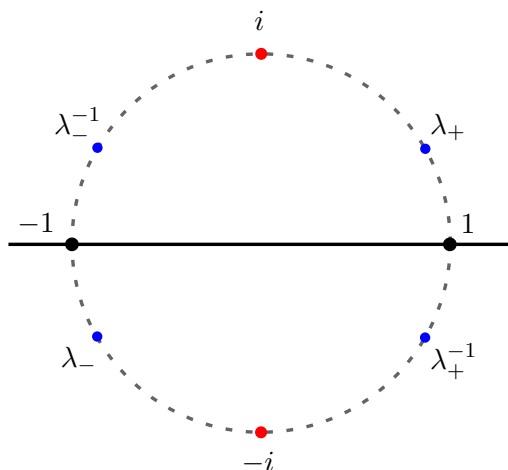


Figure 1. The four poles $\lambda_{\pm}, \lambda_{\pm}^{-1}$ of the twist function $\varphi_{\epsilon}(\lambda)$ for $\epsilon \in [0, 1]$.

Since we are interested in deforming away from the coset σ -model, we shall focus on the region $0 \leq \epsilon < 1$. We will discuss briefly what happens at the special value $\epsilon = 1$ in a moment.

Definition of g . We would now like to generalise the procedure used in the case of the principal chiral model for defining the field g at non-zero values of the deformation parameter $\epsilon \neq 0$. The novelty here is that the deformed twist function has four simple poles $\lambda_{\pm}^{\pm 1}$ and $\lambda_{\pm}^{\pm 1}$ at generic values of $\epsilon \neq 0$, which degenerate in the limit $\epsilon \rightarrow 0$ to the pair of double poles at $\lambda = \pm 1$, respectively. However, since the field g of the coset σ -model is extracted from the point $\lambda = 1$ alone, it is natural to focus only on the points $\lambda_{\pm}^{\pm 1}$ for the purpose of extracting the field g at $\epsilon \neq 0$.

Owing to the reality conditions $A^{\dagger} = -A$ and $\Pi^{\dagger} = -\Pi$ we have for the Lax matrix

$$\mathcal{L}(\lambda)^{\dagger} = (A^{(0)})^{\dagger} + \frac{1}{2}(\bar{\lambda}^{-1} + \bar{\lambda})(A^{(1)})^{\dagger} + \frac{1}{2}(1 - \bar{\lambda}^2)(\Pi^{(0)})^{\dagger} + \frac{1}{2}(\bar{\lambda}^{-1} - \bar{\lambda})(\Pi^{(1)})^{\dagger} = -\mathcal{L}(\bar{\lambda}). \quad (3.12)$$

In particular, this means that $\mathcal{L}(\lambda_{+})^{\dagger} = -\mathcal{L}(\lambda_{+}^{-1})$. By the exact same reasoning as in section 2.3 we may argue here the existence of a field $g \in F$ with the property that the gauge transformation of the Lax matrix $\mathcal{L}^g(\lambda) := \partial_{\sigma} g g^{-1} + g \mathcal{L}(\lambda) g^{-1}$ satisfies

$$\begin{aligned} (i) \quad & \mathcal{L}^g(\lambda_{\pm}^{\pm 1}) \in \mathfrak{b}^{\pm}, \\ (ii) \quad & \mathcal{L}^g(\lambda_{+})|_{\mathfrak{h}} = -\mathcal{L}^g(\lambda_{+}^{-1})|_{\mathfrak{h}}. \end{aligned} \quad (3.13)$$

The field g so defined has the required property that it reduces to the field of the coset σ -model in the limit $\epsilon \rightarrow 0$. Indeed, in this limit the pair of points $\lambda_{\pm}^{\pm 1}$ degenerate to the single point $\lambda = 1$ so that the properties (i) and (ii) together imply that $\mathcal{L}^g(1) = 0$, which is the defining condition of the coset σ -model field.

Definition of the conjugate momentum. Next, we define a field X taking values in \mathfrak{f} which will play the role of the conjugate momentum of g . In exact analogy with the

principal chiral case, we define this field as

$$X = \frac{i}{2\gamma}(\mathcal{L}^g(\lambda_+) - \mathcal{L}^g(\lambda_+^{-1})), \quad (3.14)$$

where γ is a real normalisation, the dependence of which on the deformation parameter ϵ will be fixed later. The reality condition (3.12) on the Lax matrix leads to $X^\dagger = -X$, therefore ensuring that the field X takes values in \mathfrak{f} , as desired.

Introducing the same non-split R -matrix as in (2.22) we may then also invert the relation (3.14) to express the value of the Lax matrix at the points $\lambda_\pm^{\pm 1}$ explicitly as

$$\mathcal{L}^g(\lambda_\pm^{\pm 1}) = \gamma(R \mp i)X. \quad (3.15)$$

Behaviour at $\epsilon = 1$. It turns out that the deformation of the coset σ -model that we consider here will only be valid in the range $0 \leq \epsilon < 1$. To understand what happens at $\epsilon = 1$, note that the poles of the twist function meet again in pairs $\lambda = \pm i$. In a neighbourhood of the point $\lambda = i$ the Lax matrix (3.3) reads

$$\mathcal{L}(\lambda) = A^{(0)} + \Pi^{(0)} - i\Pi^{(1)} - i(\lambda - i)(\Pi^{(0)} + iA^{(1)}) + O((\lambda - i)^2). \quad (3.16)$$

Using the deformed Poisson bracket given in appendix D one easily checks that the quantities

$$\hat{A} = A^{(0)} + \Pi^{(0)} - i\Pi^{(1)}, \quad \hat{\Pi} = \Pi^{(0)} + iA^{(1)}$$

have Poisson brackets at $\epsilon = 1$ which are identical to the undeformed Poisson brackets of the coset σ -model, namely

$$\begin{aligned} \{\hat{A}_1(\sigma), \hat{A}_2(\sigma')\}_1 &= 0, & \{\hat{\Pi}_1(\sigma), \hat{\Pi}_2(\sigma')\}_1 &= [C_{12}, \hat{\Pi}_2(\sigma)]\delta_{\sigma\sigma'}, \\ \{\hat{\Pi}_1(\sigma), \hat{A}_2(\sigma')\}_1 &= [C_{12}, \hat{A}_2(\sigma)]\delta_{\sigma\sigma'} - C_{12}\delta'_{\sigma\sigma'}. \end{aligned}$$

Notice that equation (3.16) is then completely analogous to equation (3.5) which gave the expansion of the Lax matrix around $\lambda = 1$. One can show that the model at $\epsilon = 1$ corresponds again to an undeformed coset σ -model. However, its fields $(\hat{A}, \hat{\Pi})$ no longer take values in the compact real form \mathfrak{f} , but instead satisfy the modified reality condition $\hat{A}^\dagger = -\sigma(\hat{A})$, $\hat{\Pi}^\dagger = -\sigma(\hat{\Pi})$. In this case, the group valued field should no longer be taken in the compact Lie group F .

3.4 The deformed model

In order to describe the dynamics of the Hamiltonian fields (g, X) we need to relate these to the fields $(A^{(0,1)}, \Pi^{(0,1)})$ used up until now. This is done by expressing the Lax matrix at the points $\lambda_\pm^{\pm 1} = e^{\pm i\theta}$ in two separate ways. On the one hand, the definition of the fields (g, X) enable us to write the gauge transformation of the Lax matrix with parameter $g \in F$ in terms of the field $X \in \mathfrak{f}$. Specifically, we have

$$\mathcal{L}(e^{\pm i\theta}) = -g^{-1}\partial_\sigma g + \gamma g^{-1}((R \mp i)X)g. \quad (3.17)$$

On the other hand, the value of the Lax matrix at these points may also be determined directly from its definition as

$$\mathcal{L}(e^{\pm i\theta}) = A^{(0)} + \cos\theta A^{(1)} \mp ie^{\pm i\theta} \sin\theta \Pi^{(0)} \mp i \sin\theta \Pi^{(1)}. \quad (3.18)$$

Therefore, equating the two expressions (3.17) and (3.18) we find

$$A^{(0)} + \cos\theta A^{(1)} \mp ie^{\pm i\theta} \sin\theta \Pi^{(0)} \mp i \sin\theta \Pi^{(1)} = -g^{-1} \partial_\sigma g + \gamma g^{-1} ((R \mp i)X)g.$$

Taking the sum and the difference of both sides then yields

$$\begin{aligned} A^{(0)} + \cos\theta A^{(1)} + \sin^2\theta \Pi^{(0)} &= -g^{-1} \partial_\sigma g + \gamma g^{-1} (RX)g, \\ \sin\theta \Pi^{(1)} + \cos\theta \sin\theta \Pi^{(0)} &= \gamma g^{-1} Xg. \end{aligned}$$

To extract the individual fields $A^{(0,1)}$ and $\Pi^{(0,1)}$ from these expressions we should project onto the graded subspaces $\mathfrak{f}^{(0)}$ and $\mathfrak{f}^{(1)}$ of the Lie algebra \mathfrak{f} using the corresponding projection operators P_0 and P_1 . This gives

$$A^{(0)} = P_0(-g^{-1} \partial_\sigma g + \gamma g^{-1} ((R - \eta)X)g), \quad (3.19a)$$

$$A^{(1)} = \sqrt{1 + \eta^2} P_1(-g^{-1} \partial_\sigma g + \gamma g^{-1} (RX)g), \quad (3.19b)$$

$$\Pi^{(0)} = \gamma \eta^{-1} (1 + \eta^2) P_0(g^{-1} Xg), \quad (3.19c)$$

$$\Pi^{(1)} = \gamma \eta^{-1} \sqrt{1 + \eta^2} P_1(g^{-1} Xg), \quad (3.19d)$$

where we have defined the variable

$$\eta = \tan\theta = \frac{\epsilon}{\sqrt{1 - \epsilon^2}}.$$

Quite remarkably, one can check that these expressions satisfy the deformed Poisson algebra given in appendix D exactly, if we let

$$\gamma = -\epsilon \sqrt{1 - \epsilon^2} = -\frac{\eta}{1 + \eta^2} \quad (3.20)$$

and require the fields X and g to satisfy the exact same Poisson bracket relations as in the principal chiral model, namely (2.27).

3.5 Deformed coset σ -model action

In this section, we perform the inverse Legendre transform to derive the action corresponding to our model.

Lagrangian. The analysis proceeds in exactly the same way as in subsection 2.6, except for the fact that there is now a constraint. We start with the definition of the inverse Legendre transform

$$L = \kappa(\partial_\tau g g^{-1}, X) - h^\epsilon = \kappa((g^{-1} \partial_\tau g)^{(1)}, (g^{-1} Xg)^{(1)}) - T_{++} - T_{--}. \quad (3.21)$$

Here we have used equations (3.9) and (3.2). Furthermore, we have imposed the constraint $\Pi^{(0)} \simeq 0$ and made use of its explicit expression (3.19c).

In order to relate the field X to $(g^{-1}\partial_\tau g)^{(1)}$, we compute the time evolution of g . First of all, the fields $A_\pm^{(1)}$ entering the expression (3.2) for the Hamiltonian can be written in terms of X and g using the relations (3.19),

$$A_\pm^{(1)} = \frac{-1}{\sqrt{1+\eta^2}} P_1(g^{-1}Xg \mp \eta g^{-1}RXg) \pm \sqrt{1+\eta^2}(g^{-1}\partial_\sigma g)^{(1)}. \quad (3.22)$$

This allows us to express the Hamiltonian (3.9) in terms of the fields g and X . We may then compute the time evolution of the field g as $g^{-1}\partial_\tau g = g^{-1}\{H^\epsilon, g\}_\epsilon$. Extracting the field X from this we find

$$g^{-1}Xg \simeq (g^{-1}Xg)^{(1)} = -\frac{1+\eta^2}{2} \left(\frac{1}{1-\eta P_1 \circ R_g} (g^{-1}\partial_- g)^{(1)} + \frac{1}{1+\eta P_1 \circ R_g} (g^{-1}\partial_+ g)^{(1)} \right). \quad (3.23)$$

Here we have made use once again of the constraint $\Pi^{(0)} \simeq 0$. We have also introduced the operator

$$R_g := \text{Ad } g^{-1} \circ R \circ \text{Ad } g,$$

which, like R itself, is a non-split solution of the mCYBE (2.24). Note that $1 \pm \eta P_1 \circ R_g$ is invertible on $\mathfrak{f}^{(1)}$ since it is equal to $1 \pm \eta P_1 \circ R_g \circ P_1$ and $P_1 \circ R_g \circ P_1$ is skew-symmetric.

Next, we should also eliminate the field X from $T_{\pm\pm}$ in favour of the Lagrangian field $g^{-1}\partial_\tau g$. For this, we first combine equations (3.22) and (3.23) to get

$$A_\pm^{(1)} = \sqrt{1+\eta^2} \frac{1}{1 \pm \eta P_1 \circ R_g} (g^{-1}\partial_\pm g)^{(1)}. \quad (3.24)$$

Then, plugging equations (3.23) and (3.24) in the inverse Legendre transform (3.21) yields

$$L = -\frac{1}{2}\kappa \left((g^{-1}\partial_+ g)^{(1)}, \frac{1+\eta^2}{1-\eta R_g \circ P_1} (g^{-1}\partial_- g)^{(1)} \right). \quad (3.25)$$

In the limit $\epsilon \rightarrow 0$, which corresponds to $\eta \rightarrow 0$, one correctly recovers the usual Lagrangian of the F/G coset σ -model.

Gauge invariance and field equations. One can check the gauge invariance of the deformed model directly at Lagrangian level. Indeed, under the transformation

$$g(\tau, \sigma) \mapsto g(\tau, \sigma)h(\tau, \sigma), \quad h(\tau, \sigma) \in G. \quad (3.26)$$

one has the following

$$(g^{-1}\partial_\pm g)^{(1)} \mapsto \text{Ad}(h)^{-1}(g^{-1}\partial_\pm g)^{(1)}, \quad R_g \mapsto \text{Ad}(h)^{-1} \circ R_g \circ \text{Ad}(h).$$

The gauge invariance of the action corresponding to (3.25) under (3.26) immediately follows from this. In particular, for all values of the deformation parameter η , the physical degrees of freedom belong to the coset F/G . One may also check that the field equations take the same form as in the coset σ -model, that is

$$\left(\partial_+ B_-^{(1)} + [B_+^{(0)}, B_-^{(1)}] \right) + \left(\partial_- B_+^{(1)} + [B_-^{(0)}, B_+^{(1)}] \right) = 0,$$

where the fields B_{\pm} are deformations of $g^{-1}\partial_{\pm}g$ defined as¹

$$B_{\pm} = \frac{1}{1 \pm \eta R_g \circ P_1} (g^{-1}\partial_{\pm}g). \tag{3.27}$$

Moreover, provided the field equations are satisfied, the fields $A_{\pm} = B_{\pm}^{(0)} + \sqrt{1 + \eta^2} B_{\pm}^{(1)}$ satisfy the zero curvature equation

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0. \tag{3.28}$$

The Lax pair associated with the model just defined may therefore be written as

$$\mathcal{L}_{\pm}(\lambda) = A_{\pm}^{(0)} + \lambda^{\pm 1} A_{\pm}^{(1)}.$$

3.6 Symmetry algebra

To end this section we discuss the effect of the deformation on the global F_L symmetry of the coset σ -model.

Recall that in the case of the principal chiral model, the derivation of the Poisson algebra (2.41), (2.43) and the reality conditions (2.44) satisfied by the generators of the deformed F_L symmetry relied solely on the Poisson bracket of the field X with itself in (2.27b), along with the special form (2.25) of the Lax matrix at the pair of poles λ_{\pm} of the twist function. The situation in the present case is exactly the same since the Poisson brackets (2.27) are identical and the Lax matrix at the special points $\lambda_{\pm}^{\pm 1}$ takes the similar form (3.15). The analysis therefore goes through unchanged in the case at hand, the only difference being the dependence of the parameter γ on ϵ , resulting in a different expression for q . Note also that the corresponding charges are gauge invariant. This is so because they are built in terms of X , which has vanishing Poisson bracket with $\Pi^{(0)}$.

The deformed coset σ -model therefore admits a classical $U_q^P(\mathfrak{f})$ symmetry where the parameter q is now given by

$$q = e^{\gamma} = \exp\left(-\epsilon\sqrt{1 - \epsilon^2}\right).$$

4 Deformed $SU(2)/U(1)$ coset σ -model

As recalled in the introduction, the Lagrangian of the Yang-Baxter σ -model (2.57) on a compact Lie group F reduces in the special case of $F = SU(2)$ to that of the squashed sphere σ -model. As its name suggests, the target space of the latter is a certain deformation of the 3-sphere $SU(2) \simeq S^3$. More generally, however, the deformation is not purely metric since the presence of the R -matrix in the Lagrangian gives rise to a torsion term as well [5].

For similar reasons, the Lagrangian (3.25) of the deformed coset σ -model will correspond not only to a deformation of the metric of the coset F/G , but also to the introduction of torsion in the deformed geometry. In the present section we consider the Lagrangian (3.25) in the simplest case, which corresponds to the symmetric space $SU(2)/U(1)$. In this example, since the coset is two dimensional there is no torsion.

¹Using the fact that $1 \pm \eta P_1 \circ R_g$ is invertible on $\mathfrak{f}^{(1)}$ it follows that $1 \pm \eta R_g \circ P_1$ is invertible on \mathfrak{f} . Explicitly we have $\frac{1}{1 \pm \eta R_g \circ P_1} = P_0 + (1 \mp \eta P_0 \circ R_g) \frac{1}{1 \pm \eta P_1 \circ R_g} P_1$.

Gauge fixed action and equations of motion. To begin with we set up some notation. We write the field $g \in \text{SU}(2)$ explicitly as

$$g = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \quad |z_1|^2 + |z_2|^2 = 1.$$

Correspondingly, we write a generic element M in the Lie algebra $\mathfrak{su}(2)$ as

$$M = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & -\alpha \end{pmatrix}, \quad \bar{\alpha} = -\alpha. \tag{4.1}$$

The anti-linear anti-involution $M \mapsto M^\dagger$ is defined here in terms of conjugation and matrix transposition as $M^\dagger = \bar{M}^\top$. The basis generators (A.4) are proportional to the Pauli matrices, explicitly $T = i\sigma_3$, $B = i\sigma_1/\sqrt{2}$ and $C = i\sigma_2/\sqrt{2}$. The \mathbb{Z}_2 -automorphism σ of $\mathfrak{su}(2)$ is taken to be $\sigma(M) = -\sigma_1 M^\top \sigma_1$, so that the projectors onto the grade 0 and grade 1 parts of M are respectively given by

$$P_0 M = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad P_1 M = \begin{pmatrix} 0 & -\bar{\beta} \\ \beta & 0 \end{pmatrix}.$$

Finally, the action of the R -matrix defined in (2.22) on the generic element (4.1) of $\mathfrak{su}(2)$ reads

$$R M = \begin{pmatrix} 0 & i\bar{\beta} \\ i\beta & 0 \end{pmatrix}.$$

To evaluate the Lagrangian (3.25) more explicitly we need to invert the operator $1 - \eta R_g \circ P_1$. A short calculation leads to

$$\frac{1}{1 - \eta R_g \circ P_1} M = \begin{pmatrix} \alpha' & -\bar{\beta}' \\ \beta' & -\alpha' \end{pmatrix},$$

$$\beta' = \frac{\beta}{1 - i\eta(|z_1|^2 - |z_2|^2)}, \quad \alpha' = \alpha + i\eta(\beta' \bar{z}_1 \bar{z}_2 + \bar{\beta}' z_1 z_2).$$

In the case at hand, the model described by the Lagrangian (3.25) is invariant under the right $U(1)$ gauge transformations

$$g(\sigma, \tau) \mapsto g(\sigma, \tau) \begin{pmatrix} e^{i\theta(\sigma, \tau)} & 0 \\ 0 & e^{-i\theta(\sigma, \tau)} \end{pmatrix}.$$

We choose to fix this gauge invariance by requiring the component field $z_1(\sigma, \tau)$ to be real and positive and parameterise the remaining fields using stereographic coordinates on the sphere. Hence, we take

$$g = \frac{1}{\sqrt{1 + \bar{\psi}\psi}} \begin{pmatrix} 1 & -\bar{\psi} \\ \psi & 1 \end{pmatrix}.$$

In terms of the complex field ψ , the action associated with the Lagrangian (3.25) then takes the form

$$S[\psi, \bar{\psi}] = \frac{1 + \eta^2}{2} \int d\tau d\sigma \frac{\partial_- \psi \partial_+ \bar{\psi} + \partial_+ \psi \partial_- \bar{\psi}}{(1 + \bar{\psi}\psi)^2 + \eta^2(1 - \bar{\psi}\psi)^2}. \quad (4.2)$$

A term which is skew-symmetric in the light-cone coordinates has been omitted here, since it does not participate in the field equation, which reads

$$\partial_+ \partial_- \psi - 2 \frac{1 + \bar{\psi}\psi - \eta^2(1 - \bar{\psi}\psi)}{(1 + \bar{\psi}\psi)^2 + \eta^2(1 - \bar{\psi}\psi)^2} \bar{\psi} \partial_+ \psi \partial_- \psi = 0. \quad (4.3)$$

Zero curvature equation. In the chosen gauge, we find that the fields (3.27) entering the equations of motion are given by

$$B_{\pm} = \begin{pmatrix} a_{\pm} & -\bar{b}_{\pm} \\ b_{\pm} & -a_{\pm} \end{pmatrix},$$

$$b_{\pm} = \frac{\partial_{\pm} \psi}{1 + \bar{\psi}\psi \pm i\eta(1 - \bar{\psi}\psi)}, \quad a_{\pm} = \frac{1}{2}(1 \mp i\eta)\bar{\psi}b_{\pm} - \frac{1}{2}(1 \pm i\eta)\psi\bar{b}_{\pm}.$$

In terms of these quantities, the field equation (4.3) reduces to the covariant conservation equation

$$(\partial_+ b_- - 2a_+ b_-) + (\partial_- b_+ - 2a_- b_+) = 0.$$

Moreover, provided the field equation is satisfied, one has

$$\begin{aligned} (\partial_+ b_- - 2a_+ b_-) - (\partial_- b_+ - 2a_- b_+) &= 0, \\ \partial_+ a_- - \partial_- a_+ - (1 + \eta^2)(\bar{b}_+ b_- - \bar{b}_- b_+) &= 0, \end{aligned}$$

corresponding to the projections on the two gradings of the zero curvature equation (3.28).

Remarks. The action (4.2) has the following interesting property. It interpolates between the coset σ -model on the compact symmetric space $SU(2)/U(1)$ at $\eta = 0$ and the coset σ -model on the non-compact symmetric space $SU(1, 1)/U(1)$ at $\eta = \infty$. This is reminiscent of the discussion at the end of the subsection 3.3. Indeed, the limit $\eta \rightarrow \infty$ corresponds to $\epsilon \rightarrow 1$ and we have shown that at this special point, the model constructed corresponds to an undeformed coset σ -model.

We end this section by computing, for generic values of η , the Ricci tensor associated with the metric g_{ij} appearing in the action (4.2). Its only non-vanishing component is given by

$$\begin{aligned} R_{\psi\bar{\psi}} &= \frac{\partial}{\partial\psi} \frac{\partial}{\partial\bar{\psi}} \ln((1 + \bar{\psi}\psi)^2 + \eta^2(1 - \bar{\psi}\psi)^2) \\ &= \frac{2(1 - \eta^2)}{(1 + \bar{\psi}\psi)^2 + \eta^2(1 - \bar{\psi}\psi)^2} + \frac{16\eta^2\bar{\psi}\psi}{((1 + \bar{\psi}\psi)^2 + \eta^2(1 - \bar{\psi}\psi)^2)^2}. \end{aligned} \quad (4.4)$$

The second term in (4.4) vanishes in both limits $\eta \rightarrow 0$ and $\eta \rightarrow \infty$, at which we have $R_{ij} = \pm 4g_{ij}$ respectively. It is only in these two limits that one recovers an Einstein manifold, with opposite curvatures. It is well-known that the on-shell one-loop divergence in such a model is proportional to the Ricci tensor [21, 22]. In the case at hand, such a divergence can be reabsorbed into a renormalization of an overall factor in front of the action as in the coset σ -model case and into a renormalization of the deformation parameter η .

5 Conclusion

In this article we introduced a procedure for constructing integrable deformations of principal chiral models and symmetric space σ -models associated with compact Lie groups. It is worth emphasising that in this construction, the integrability of the deformed models is obvious from the very outset. Indeed, the deformation originates from the choice of a second Poisson bracket which is compatible with the original one. As such, the generalized Faddeev-Reshetikhin bracket plays an essential role in the initial step of the construction. As in the case of the anisotropic $SU(2)$ principal chiral model, a natural question to consider is whether two-parameter deformations of these σ -models may also be constructed within this framework using a third compatible Poisson bracket.

Another important ingredient is given by the non-split R -matrix which shows up in the resulting Lagrangians. In fact, the integrability of the corresponding field equations relies in a subtle way on the modified classical Yang-Baxter equation for this R -matrix. Its appearance in our construction can be traced back to the fact that the gauge transformed Lax matrix takes values in Borel subalgebras at the poles of the twist function. Moreover, this latter property was essential in order to extract the classical q -deformed $U_q^{\mathcal{P}}(\mathfrak{f})$ symmetry algebra.

The charges associated with the q -deformed $U_q(\mathfrak{f})$ symmetry were extracted from the leading order behaviour of the monodromy matrix at the poles of the twist function. This raises a natural question with regards to the higher conserved charges. By extracting these from the higher order expansion of the gauge transformed monodromy matrix at the poles of the twist function, we may anticipate that they should satisfy a classical affine $U_q^{\mathcal{P}}(\widehat{\mathfrak{f}})$ Poisson-Hopf algebra. Indeed, in the case of the squashed sphere σ -model, the hidden symmetries were already shown to satisfy a $U_q^{\mathcal{P}}(\widehat{\mathfrak{sl}}_2)$ algebra [14].

Much like the squashed sphere σ -model, the example of the deformed $SU(2)/U(1)$ coset σ -model is simple enough that it can be studied very explicitly. In fact, many of the general properties discussed in the general case are also present in this simplest example. This integrable deformation certainly deserves further study.

It is very exciting to consider the possible generalisation of this work. The case of the $AdS_5 \times S^5$ superstring σ -model, currently under investigation, is particularly enticing, especially because the generalisation of the Faddeev-Reshetikhin Poisson bracket is already known [23].

A Compact real form

Let F be a compact Lie group with Lie algebra $\mathfrak{f} = \text{Lie}(F)$. We denote by $\mathfrak{f}^{\mathbb{C}}$ the complexification of \mathfrak{f} and fix a choice of Cartan subalgebra \mathfrak{h} with corresponding root space decomposition

$$\mathfrak{f}^{\mathbb{C}} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathbb{C}E^{\alpha} \right).$$

Given a choice of simple roots $\alpha_i \in \Phi$, $i = 1, \dots, n = \text{rk } \mathfrak{f}^{\mathbb{C}}$ we denote the pair of opposite nilpotent subalgebras as $\mathfrak{n}^{\pm} = \bigoplus_{\alpha > 0} \mathbb{C}E^{\pm\alpha}$ and the corresponding Borel subalgebras as

$\mathfrak{b}^\pm = \mathfrak{h} \oplus \mathfrak{n}^\pm$. The non-trivial Lie algebra relations in $\mathfrak{f}^\mathbb{C}$ read, for any roots $\alpha, \beta \in \Phi$,

$$[H, E^\alpha] = \alpha(H)E^\alpha, \quad [E^\alpha, E^{-\alpha}] = H^\alpha, \quad [E^\alpha, E^\beta] = N_{\alpha,\beta}E^{\alpha+\beta}, \quad \text{if } \alpha + \beta \in \Phi$$

where $H^\alpha \in \mathfrak{h}$ is defined for any root $\alpha \in \Phi$ in terms of the Killing form on $\mathfrak{f}^\mathbb{C}$ as $\kappa(H^\alpha, H) = \alpha(H)$. The latter induces a (positive definite) inner product on the set of roots denoted $(\alpha, \beta) = \alpha(H^\beta)$. We have chosen the normalisation of the generators E^α so that

$$\kappa(E^\alpha, E^\beta) = \delta_{\alpha,-\beta}.$$

Letting $H^i = H^{\alpha_i}$ for any simple root α_i we have

$$\kappa(H^i, H^j) = \alpha_i(H^j) = (\alpha_i, \alpha_j) = B_{ij},$$

where $B_{ij} = d_i A_{ij}$ denotes the symmetrised Cartan matrix with $d_i = (\alpha_i, \alpha_i)/2$. With respect to the basis $H^i, i = 1, \dots, n$ and $E^\alpha, \alpha \in \Phi$ of $\mathfrak{f}^\mathbb{C}$, the tensor Casimir then reads

$$C_{\underline{\mathbf{12}}} = \sum_{i,j=1}^n B_{ij}^{-1} H^i \otimes H^j + \sum_{\alpha > 0} (E^\alpha \otimes E^{-\alpha} + E^{-\alpha} \otimes E^\alpha). \quad (\text{A.1})$$

If $\beta + \mathfrak{p}\alpha, \dots, \beta, \dots, \beta + \mathfrak{q}\alpha$ denotes the α -string through β , where $\mathfrak{p} \leq 0$ and $\mathfrak{q} \geq 0$, then with the above conventions one may show that

$$N_{\alpha,\beta}^2 = \mathfrak{q}(1 - \mathfrak{p}) \frac{(\alpha, \alpha)}{2}, \quad \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = -(\mathfrak{p} + \mathfrak{q}). \quad (\text{A.2})$$

In particular, the structure constants $N_{\alpha,\beta}$ are all real.

The real Lie algebra \mathfrak{f} is recovered from its complexification $\mathfrak{f}^\mathbb{C}$ as the fixed point set of a certain anti-linear involutive automorphism τ , namely such that

$$\tau(\lambda X + \mu Y) = \bar{\lambda} \tau(X) + \bar{\mu} \tau(Y), \quad \tau^2 = 1, \quad \tau([X, Y]) = [\tau(X), \tau(Y)],$$

for any $X, Y \in \mathfrak{f}^\mathbb{C}$ and $\lambda, \mu \in \mathbb{C}$. It is convenient to define $\tau(X) = -X^\dagger$ in terms of an anti-linear involutive anti-automorphism $X \mapsto X^\dagger$ with the properties

$$(\lambda X + \mu Y)^\dagger = \bar{\lambda} X^\dagger + \bar{\mu} Y^\dagger, \quad (X^\dagger)^\dagger = X, \quad [X, Y]^\dagger = [Y^\dagger, X^\dagger].$$

In the case of the compact real form we define the latter on the basis H^i, E^α as

$$(H^i)^\dagger = H^i, \quad (E^\alpha)^\dagger = E^{-\alpha}. \quad (\text{A.3})$$

We then have by definition $\mathfrak{f} = \{X \in \mathfrak{f}^\mathbb{C} \mid \tau(X) = X\}$. A basis over \mathbb{R} for the compact real form \mathfrak{f} is then given by

$$T^i = iH^i, \quad B^\alpha = \frac{i}{\sqrt{2}}(E^\alpha + E^{-\alpha}), \quad C^\alpha = \frac{1}{\sqrt{2}}(E^\alpha - E^{-\alpha}). \quad (\text{A.4})$$

With respect to these generators the Killing form reads

$$\kappa(T^i, T^j) = -B_{ij}, \quad \kappa(B^\alpha, B^\beta) = -\delta_{\alpha,-\beta}, \quad \kappa(C^\alpha, C^\beta) = -\delta_{\alpha,-\beta} \quad (\text{A.5})$$

so that the tensor Casimir may be expressed as

$$C_{\underline{\mathbf{12}}} = - \sum_{i,j=1}^n B_{ij}^{-1} T^i \otimes T^j - \sum_{\alpha > 0} (B^\alpha \otimes B^\alpha + C^\alpha \otimes C^\alpha). \quad (\text{A.6})$$

Iwasawa decomposition. Let \mathfrak{h}_0 denote the linear span over \mathbb{R} of the set of Cartan generators H^i , $i = 1, \dots, n$. Then the lower Borel subalgebra $\mathfrak{b}^- \in \mathfrak{f}^{\mathbb{C}}$ is contained in $\mathfrak{f} \oplus \mathfrak{h}_0 \oplus \mathfrak{n}^+$. Indeed, any element in \mathfrak{b}^- takes the form $X + h$ where $X = \sum_{\alpha > 0} x_\alpha E^{-\alpha} \in \mathfrak{n}^-$ and $h = \sum_{i=1}^n a_i H^i$ for some $x_\alpha, a_i \in \mathbb{C}$. It then follows using (A.3) that $X^\dagger = \sum_{\alpha > 0} \bar{x}_\alpha E^\alpha \in \mathfrak{n}^+$ and hence

$$X + h = \left(\left(X + \frac{1}{2}h \right) - \left(X + \frac{1}{2}h \right)^\dagger \right) + \frac{1}{2}(h + h^\dagger) + X^\dagger \in \mathfrak{f} \oplus \mathfrak{h}_0 \oplus \mathfrak{n}^+.$$

In particular, using the decomposition $\mathfrak{f}^{\mathbb{C}} = \mathfrak{b}^- \oplus \mathfrak{n}^+$ it follows that

$$\mathfrak{f}^{\mathbb{C}} = \mathfrak{f} \oplus \mathfrak{h}_0 \oplus \mathfrak{n}^+. \quad (\text{A.7})$$

This is known as the Iwasawa decomposition of the complex Lie algebra $\mathfrak{f}^{\mathbb{C}}$.

B q -Poisson-Serre relations

In this appendix we prove the q -Poisson-Serre relations (2.43). To do this we will define charges associated also with non-simple roots $\alpha \in \Phi^+$. This in turn requires choosing a normal ordering on the set of positive roots Φ^+ of $\mathfrak{f}^{\mathbb{C}}$ (see for instance [15, 24, 25]), namely such that if $\alpha < \beta$ and $\alpha + \beta$ is a root then $\alpha < \alpha + \beta < \beta$. Given such a choice of ordering, we write the nilpotent part of the monodromy matrix (2.33a) as follows

$$P_{\text{exp}}^< \left[\gamma \sum_{\alpha > 0} \int_{-\infty}^{\infty} d\sigma \mathfrak{J}_\alpha^E(\sigma) E^\alpha \right] = \prod_{\alpha > 0}^< \exp \left(\gamma \int_{-\infty}^{\infty} d\sigma \mathfrak{Q}_\alpha^E(\sigma) E^\alpha \right),$$

where the superscript $<$ on the product indicates the use of normal ordering on the positive roots. Note that the normal ordering only defines a partial ordering on the set of positive roots. However, whenever two roots α and β are not ordered this implies that $\alpha + \beta$ is not a root. It follows that the corresponding generators E^α and E^β commute and therefore their relative order in the above product is irrelevant.

Let α_i, α_j be simple roots and consider the collection of roots α belonging to the α_i -string through α_j , namely $\alpha_j, \alpha_j + \alpha_i, \dots, \alpha_j + \mathfrak{q} \alpha_i$ for some $\mathfrak{q} \geq 0$ such that $\alpha_j + (\mathfrak{q} + 1) \alpha_i$ is not a root. It is easy to see that for the simple root α_j we have

$$\mathfrak{Q}_{\alpha_j}^E(\sigma) = \mathfrak{J}_{\alpha_j}^E(\sigma).$$

Next, consider the sum $\alpha_j + \alpha_i$. Assuming this is a root, which is the case if $\mathfrak{q} \geq 1$, we must have either $\alpha_j < \alpha_i$ or $\alpha_i < \alpha_j$. It will be more convenient to work with a normal ordering such that $\alpha_j < \alpha_i$. In this case it follows that the roots of the α_i -string through α_j are ordered as

$$\alpha_j < \alpha_j + \alpha_i < \alpha_j + 2\alpha_i < \dots < \alpha_j + \mathfrak{q} \alpha_i < \alpha_i.$$

The charge density corresponding to the sum of simple roots $\alpha_j + \alpha_i$ is found to be

$$\mathfrak{Q}_{\alpha_j + \alpha_i}^E(\sigma) = \mathfrak{J}_{\alpha_j + \alpha_i}^E(\sigma) - \gamma N_{\alpha_j, \alpha_i} \mathfrak{J}_{\alpha_i}^E(\sigma) \int_{-\infty}^{\sigma} d\sigma' \mathfrak{J}_{\alpha_j}^E(\sigma'). \quad (\text{B.1})$$

More generally, the charge density $\mathfrak{Q}_{\alpha_j+r\alpha_i}^E(\sigma)$ associated with the root $\alpha_j+r\alpha_i$ with $0 < r \leq \mathfrak{q}$ may be expressed recursively in terms of the preceding charge density $\mathfrak{Q}_{\alpha_j+(r-1)\alpha_i}^E(\sigma)$ as follows

$$\mathfrak{Q}_{\alpha_j+r\alpha_i}^E(\sigma) = \mathfrak{J}_{\alpha_j+r\alpha_i}^E(\sigma) - \gamma N_{\alpha_j+(r-1)\alpha_i, \alpha_i} \mathfrak{J}_{\alpha_i}^E(\sigma) \int_{-\infty}^{\sigma} d\sigma' \mathfrak{Q}_{\alpha_j+(r-1)\alpha_i}^E(\sigma'). \quad (\text{B.2})$$

Finally, recalling the notation (2.40) we define the charges corresponding to each root $\alpha_j+r\alpha_i$ as

$$Q_{\alpha_j+r\alpha_i}^E = D_j D_i^r \int_{-\infty}^{\infty} d\sigma \mathfrak{Q}_{\alpha_j+r\alpha_i}^E(\sigma), \quad (\text{B.3})$$

so that in the case $r=0$ this definition agrees with (2.39).

In the remainder of this appendix we will prove that the generators defined in (B.3) satisfy the following Poisson algebra with respect to the q -Poisson bracket introduced in (2.42), for $r \leq \mathfrak{q}$,

$$\{Q_{\alpha_i}^E, Q_{\alpha_j+r\alpha_i}^E\}_{q\epsilon} = 2i N_{\alpha_j+r\alpha_i, \alpha_i} Q_{\alpha_j+(r+1)\alpha_i}^E. \quad (\text{B.4})$$

Since $\alpha_j+(\mathfrak{q}+1)\alpha_i$ is not a root by definition of \mathfrak{q} we have that $N_{\alpha_j+\mathfrak{q}\alpha_i, \alpha_i} = 0$. It therefore follows from (B.4) that

$$\underbrace{\{Q_{\alpha_i}^E, \{Q_{\alpha_i}^E, \dots \{Q_{\alpha_i}^E, Q_{\alpha_j}^E\}_{q\epsilon} \dots\}_{q\epsilon}}_{\mathfrak{q}+1 \text{ times}} = 0,$$

which is nothing but the q -Poisson-Serre relation (2.43) since $\mathfrak{q} = -A_{ij}$. In fact, to establish the q -Poisson-Serre relations for classical Lie algebras it suffices to show that (B.4) holds with $r \leq 2$ since for every pair of simple roots α_i and α_j , the α_i -string through α_j has at most $\mathfrak{q} = 2$.

Case $r=0$. We begin by proving the relation (B.4) in the case $r=0$. Comparing coefficients of E^β on both sides of the second relation in (2.37) yields

$$\{e_\alpha(\sigma), e_\beta(\sigma')\}_\epsilon = 2i N_{\beta, \alpha} e_{\alpha+\beta}(\sigma) \delta_{\sigma\sigma'}, \quad \text{if } \alpha + \beta \in \Phi.$$

Using the definition (2.34) of \mathfrak{J}_α^E this then leads to

$$\{\mathfrak{J}_\alpha^E(\sigma), \mathfrak{J}_\beta^E(\sigma')\}_\epsilon = 2i N_{\beta, \alpha} \mathfrak{J}_{\alpha+\beta}^E(\sigma) \delta_{\sigma\sigma'} + i\gamma(\alpha, \beta) \mathfrak{J}_\alpha^E(\sigma) \mathfrak{J}_\beta^E(\sigma') \epsilon_{\sigma\sigma'}. \quad (\text{B.5})$$

Introducing the Heaviside step function $\theta_{\sigma\sigma'} = \frac{1}{2}(\epsilon_{\sigma\sigma'} + 1)$ we may rewrite this as

$$\begin{aligned} \{\mathfrak{J}_\alpha^E(\sigma), \mathfrak{J}_\beta^E(\sigma')\}_\epsilon + i\gamma(\alpha, \beta) \mathfrak{J}_\alpha^E(\sigma) \mathfrak{J}_\beta^E(\sigma') \\ = 2i(N_{\beta, \alpha} \mathfrak{J}_{\alpha+\beta}^E(\sigma) \delta_{\sigma\sigma'} + \gamma(\alpha, \beta) \mathfrak{J}_\alpha^E(\sigma) \mathfrak{J}_\beta^E(\sigma') \theta_{\sigma\sigma'}). \end{aligned}$$

In terms of the q -Poisson bracket introduced in (2.42), it now follows from the above in the case $\alpha = \alpha_i$ and $\beta = \alpha_j$ that

$$\begin{aligned} \{Q_{\alpha_i}^E, Q_{\alpha_j}^E\}_{q\epsilon} &= \{Q_{\alpha_i}^E, Q_{\alpha_j}^E\}_\epsilon + i\gamma(\alpha_i, \alpha_j) Q_{\alpha_i}^E Q_{\alpha_j}^E \\ &= 2i D_i D_j \left(N_{\alpha_j, \alpha_i} \int_{-\infty}^{\infty} d\sigma \mathfrak{J}_{\alpha_j+\alpha_i}^E(\sigma) \right. \\ &\quad \left. + \gamma(\alpha_j, \alpha_i) \int_{-\infty}^{\infty} d\sigma \mathfrak{J}_{\alpha_i}^E(\sigma) \int_{-\infty}^{\sigma} d\sigma' \mathfrak{J}_{\alpha_j}^E(\sigma') \right). \end{aligned}$$

Now using (A.2) with $\alpha = \alpha_i$ and $\beta = \alpha_j$, since $\alpha_j - \alpha_i$ is not a root we have $\mathfrak{p} = 0$ from which we deduce that $N_{\alpha_j, \alpha_i}^2 = -(\alpha_j, \alpha_i)$. Hence we deduce using the definitions (B.1) and (B.3) that

$$\{Q_{\alpha_i}^E, Q_{\alpha_j}^E\}_{q\epsilon} = 2iN_{\alpha_j, \alpha_i}Q_{\alpha_j + \alpha_i}^E.$$

Cases $r = 1$ and $r = 2$. The relation (B.4) in the cases $r = 1$ and $r = 2$ follows in a similar way. For instance, starting from the definition (B.1) and the relation (B.5) one can show that

$$\begin{aligned} & \{\mathfrak{J}_{\alpha_i}^E(\sigma), \mathfrak{Q}_{\alpha_j + \alpha_i}^E(\sigma')\}_{\epsilon} + i\gamma(\alpha_i, \alpha_j + \alpha_i)\mathfrak{J}_{\alpha_i}^E(\sigma)\mathfrak{Q}_{\alpha_j + \alpha_i}^E(\sigma') \\ &= 2i\left(N_{\alpha_j + \alpha_i, \alpha_i}\mathfrak{J}_{\alpha_j + 2\alpha_i}^E(\sigma)\delta_{\sigma\sigma'} \right. \\ & \quad \left. + \gamma(\alpha_i, \alpha_j + \alpha_i)\mathfrak{J}_{\alpha_i}^E(\sigma)\mathfrak{Q}_{\alpha_j + \alpha_i}^E(\sigma')\theta_{\sigma\sigma'} + \gamma(\alpha_i, \alpha_j)\mathfrak{J}_{\alpha_i}^E(\sigma')\mathfrak{Q}_{\alpha_j + \alpha_i}^E(\sigma)\theta_{\sigma'\sigma}\right). \end{aligned}$$

Taking the integral over σ and σ' then yields the desired relation (B.4) in the case $r = 1$, namely

$$\{Q_{\alpha_i}^E, Q_{\alpha_j + \alpha_i}^E\}_{q\epsilon} = 2iN_{\alpha_j + \alpha_i, \alpha_i}Q_{\alpha_j + 2\alpha_i}^E.$$

In deriving these results we make use of the following useful identities, valid for any $0 \leq r \leq \mathfrak{q}$,

$$\begin{aligned} N_{\alpha_j + r\alpha_i, \alpha_i}^2 &= -\left((r+1)\alpha_j + \frac{r(r+1)}{2}\alpha_i, \alpha_i\right), \\ -N_{\alpha_j + (r-1)\alpha_i, \alpha_i}^2 + (\alpha_j + r\alpha_i, \alpha_i) &= -N_{\alpha_j + r\alpha_i, \alpha_i}^2. \end{aligned}$$

Finally, in the case $r = 2$, a lengthy calculation leads to the following

$$\begin{aligned} & \{\mathfrak{J}_{\alpha_i}^E(\sigma), \mathfrak{Q}_{\alpha_j + 2\alpha_i}^E(\sigma')\}_{\epsilon} + i\gamma(\alpha_i, \alpha_j + 2\alpha_i)\mathfrak{J}_{\alpha_i}^E(\sigma)\mathfrak{Q}_{\alpha_j + 2\alpha_i}^E(\sigma') \\ &= 2i\left(N_{\alpha_j + 2\alpha_i, \alpha_i}\mathfrak{J}_{\alpha_j + 3\alpha_i}^E(\sigma)\delta_{\sigma\sigma'} \right. \\ & \quad + \gamma(\alpha_i, \alpha_j + 2\alpha_i)\mathfrak{J}_{\alpha_i}^E(\sigma)\mathfrak{J}_{\alpha_j + 2\alpha_i}^E(\sigma')\theta_{\sigma\sigma'} - \gamma N_{\alpha_j + \alpha_i, \alpha_i}^2\mathfrak{J}_{\alpha_i}^E(\sigma')\mathfrak{J}_{\alpha_j + 2\alpha_i}^E(\sigma)\theta_{\sigma'\sigma} \\ & \quad - \gamma^2 N_{\alpha_j + \alpha_i, \alpha_i}\left((\alpha_i, \alpha_j + \alpha_i)\mathfrak{J}_{\alpha_i}^E(\sigma)\mathfrak{J}_{\alpha_i}^E(\sigma')\int_{-\infty}^{\sigma'} d\sigma''\mathfrak{Q}_{\alpha_j + \alpha_i}^E(\sigma'')\theta_{\sigma\sigma'} \right. \\ & \quad \left. + (\alpha_i, \alpha_j + \alpha_i)\mathfrak{J}_{\alpha_i}^E(\sigma)\mathfrak{J}_{\alpha_i}^E(\sigma')\int_{-\infty}^{\sigma} d\sigma''\mathfrak{Q}_{\alpha_j + \alpha_i}^E(\sigma'')\theta_{\sigma'\sigma} \right. \\ & \quad \left. + (\alpha_i, \alpha_i)\mathfrak{J}_{\alpha_i}^E(\sigma)\mathfrak{J}_{\alpha_i}^E(\sigma')\int_{-\infty}^{\sigma'} d\sigma''\mathfrak{Q}_{\alpha_j + \alpha_i}^E(\sigma'')\theta_{\sigma\sigma'} \right. \\ & \quad \left. + (\alpha_i, \alpha_j)\mathfrak{J}_{\alpha_i}^E(\sigma')\mathfrak{Q}_{\alpha_j + \alpha_i}^E(\sigma)\int_{-\infty}^{\sigma'} d\sigma''\mathfrak{J}_{\alpha_i}^E(\sigma'')\theta_{\sigma'\sigma} \right. \\ & \quad \left. - (\alpha_i, \alpha_j)\mathfrak{J}_{\alpha_i}^E(\sigma')\mathfrak{Q}_{\alpha_j + \alpha_i}^E(\sigma)\int_{-\infty}^{\sigma} d\sigma''\mathfrak{J}_{\alpha_i}^E(\sigma'')\theta_{\sigma'\sigma}\right). \end{aligned}$$

After taking the integral over σ and σ' we obtain the sought after relation (B.4) with $r = 2$, namely

$$\{Q_{\alpha_i}^E, Q_{\alpha_j + 2\alpha_i}^E\}_{q\epsilon} = 2iN_{\alpha_j + 2\alpha_i, \alpha_i}Q_{\alpha_j + 3\alpha_i}^E.$$

C Modified classical Yang-Baxter equation

The modified classical Yang-Baxter equation (2.24) satisfied by the R -matrix (2.22) in the present article (see also [5]) is slightly different from the one which appeared in [7]. The general form of this equation over a real Lie algebra \mathfrak{f} reads

$$[RX, RY] - R([RX, Y] + [X, RY]) = -\omega[X, Y], \quad (\text{C.1})$$

for some real parameter $\omega \in \mathbb{R}$. Of course, by rescaling the linear map $R \in \text{End } \mathfrak{f}$ by $1/\sqrt{|\omega|}$ we may restrict attention to the cases $\omega = \pm 1$. The R -matrices discussed in [7] are solutions of this equation with $\omega = 1$, sometimes referred to as the ‘split case’. However, the R -matrix (2.22) used here and in [5] is a solution of this equation with $\omega = -1$, referred to as the ‘non-split case’.

In either case, the modified classical Yang-Baxter equation (C.1) may be rewritten as

$$(R \pm \sqrt{\omega})([X, Y]_R) = [(R \pm \sqrt{\omega})X, (R \pm \sqrt{\omega})Y], \quad (\text{C.2})$$

where $[X, Y]_R := [RX, Y] + [X, RY]$ defines a second Lie bracket on \mathfrak{f} by virtue of (C.1). In the split case, this implies that the linear maps $R_{\pm} := R \pm 1$ are both Lie algebra homomorphisms $\mathfrak{f}_R \rightarrow \mathfrak{f}$ where \mathfrak{f}_R is the vector space \mathfrak{f} equipped with the Lie bracket $[\cdot, \cdot]_R$. In the non-split case, however, things are a little more subtle. Since $\sqrt{\omega} = i$, we see that the linear maps $R \pm i$ are still Lie algebra homomorphisms by (C.2) but now from $\mathfrak{f}_R \rightarrow \mathfrak{f}^{\mathbb{C}}$. Recall that in the split case ($\omega = 1$), the pair of maps R_{\pm} can be used to define an embedding $\mathfrak{f}_R \rightarrow \mathfrak{f} \oplus \mathfrak{f}$ so that \mathfrak{f}_R may be regarded as a subalgebra of the double $\mathfrak{f} \oplus \mathfrak{f}$. In the present non-split case ($\omega = -1$), however, the map $R - i$ alone defines an embedding of the real Lie algebra \mathfrak{f}_R into the complexification $\mathfrak{f}^{\mathbb{C}}$.

D Deformed Poisson bracket for coset σ -model

The deformed Poisson bracket (3.7), when expressed in terms of the graded components of the fields A and Π , takes the following form

$$\begin{aligned} \{A_{\underline{1}}^{(0)}(\sigma), A_{\underline{2}}^{(0)}(\sigma')\}_{\epsilon} &= -\epsilon^2 [C_{\underline{12}}^{(00)}, 2A_{\underline{2}}^{(0)}(\sigma) + \Pi_{\underline{2}}^{(0)}(\sigma)] \delta_{\sigma\sigma'} + 2\epsilon^2 C_{\underline{12}}^{(00)} \delta'_{\sigma\sigma'}, \\ \{A_{\underline{1}}^{(0)}(\sigma), A_{\underline{2}}^{(1)}(\sigma')\}_{\epsilon} &= -\epsilon^2 [C_{\underline{12}}^{(00)}, A_{\underline{2}}^{(1)}(\sigma) + \Pi_{\underline{2}}^{(1)}(\sigma)] \delta_{\sigma\sigma'}, \\ \{A_{\underline{1}}^{(1)}(\sigma), A_{\underline{2}}^{(1)}(\sigma')\}_{\epsilon} &= -\epsilon^2 [C_{\underline{12}}^{(11)}, \Pi_{\underline{2}}^{(0)}(\sigma)] \delta_{\sigma\sigma'}, \\ \{A_{\underline{1}}^{(0)}(\sigma), \Pi_{\underline{2}}^{(0)}(\sigma')\}_{\epsilon} &= [C_{\underline{12}}^{(00)}, A_{\underline{2}}^{(0)}(\sigma)] \delta_{\sigma\sigma'} - C_{\underline{12}}^{(00)} \delta'_{\sigma\sigma'}, \\ \{A_{\underline{1}}^{(0)}(\sigma), \Pi_{\underline{2}}^{(1)}(\sigma')\}_{\epsilon} &= (1 - \epsilon^2) [C_{\underline{12}}^{(00)}, A_{\underline{2}}^{(1)}(\sigma)] \delta_{\sigma\sigma'} - \epsilon^2 [C_{\underline{12}}^{(00)}, \Pi_{\underline{2}}^{(1)}(\sigma)] \delta_{\sigma\sigma'}, \\ \{A_{\underline{1}}^{(1)}(\sigma), \Pi_{\underline{2}}^{(0)}(\sigma')\}_{\epsilon} &= [C_{\underline{12}}^{(11)}, A_{\underline{2}}^{(1)}(\sigma)] \delta_{\sigma\sigma'}, \\ \{A_{\underline{1}}^{(1)}(\sigma), \Pi_{\underline{2}}^{(1)}(\sigma')\}_{\epsilon} &= [C_{\underline{12}}^{(11)}, A_{\underline{2}}^{(0)}(\sigma)] \delta_{\sigma\sigma'} + \epsilon^2 [C_{\underline{12}}^{(11)}, \Pi_{\underline{2}}^{(0)}(\sigma)] \delta_{\sigma\sigma'} - C_{\underline{12}}^{(11)} \delta'_{\sigma\sigma'}, \\ \{\Pi_{\underline{1}}^{(0)}(\sigma), \Pi_{\underline{2}}^{(0)}(\sigma')\}_{\epsilon} &= [C_{\underline{12}}^{(00)}, \Pi_{\underline{2}}^{(0)}(\sigma)] \delta_{\sigma\sigma'}, \\ \{\Pi_{\underline{1}}^{(0)}(\sigma), \Pi_{\underline{2}}^{(1)}(\sigma')\}_{\epsilon} &= [C_{\underline{12}}^{(00)}, \Pi_{\underline{2}}^{(1)}(\sigma)] \delta_{\sigma\sigma'}, \\ \{\Pi_{\underline{1}}^{(1)}(\sigma), \Pi_{\underline{2}}^{(1)}(\sigma')\}_{\epsilon} &= (1 - \epsilon^2) [C_{\underline{12}}^{(11)}, \Pi_{\underline{2}}^{(0)}(\sigma)] \delta_{\sigma\sigma'}. \end{aligned}$$

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References

- [1] I. Cherednik, *Relativistically invariant quasiclassical limits of integrable two-dimensional quantum models*, *Theor. Math. Phys.* **47** (1981) 422 [INSPIRE].
- [2] I. Kawaguchi and K. Yoshida, *Hybrid classical integrability in squashed σ -models*, *Phys. Lett. B* **705** (2011) 251 [arXiv:1107.3662] [INSPIRE].
- [3] I. Kawaguchi, T. Matsumoto and K. Yoshida, *On the classical equivalence of monodromy matrices in squashed σ -model*, *JHEP* **06** (2012) 082 [arXiv:1203.3400] [INSPIRE].
- [4] C. Klimčík, *Yang-Baxter σ -models and dS/AdS T duality*, *JHEP* **12** (2002) 051 [hep-th/0210095] [INSPIRE].
- [5] C. Klimčík, *On integrability of the Yang-Baxter σ -model*, *J. Math. Phys.* **50** (2009) 043508 [arXiv:0802.3518] [INSPIRE].
- [6] L. Faddeev and N.Y. Reshetikhin, *Integrability of the principal chiral field model in $(1+1)$ -dimension*, *Annals Phys.* **167** (1986) 227 [INSPIRE].
- [7] F. Delduc, M. Magro and B. Vicedo, *Alleviating the non-ultralocality of coset σ -models through a generalized Faddeev-Reshetikhin procedure*, *JHEP* **08** (2012) 019 [arXiv:1204.0766] [INSPIRE].
- [8] J.M. Maillet, *Kac-Moody algebra and extended Yang-Baxter relations in the $O(N)$ nonlinear σ model*, *Phys. Lett. B* **162** (1985) 137 [INSPIRE].
- [9] J.M. Maillet, *New integrable canonical structures in two-dimensional models*, *Nucl. Phys. B* **269** (1986) 54 [INSPIRE].
- [10] A. Reyman and M. Semenov-Tian-Shansky, *Compatible Poisson structures for Lax equations: an R matrix approach*, *Phys. Lett. A* **130** (1988) 456 [INSPIRE].
- [11] O. Babelon, *Jimbo's q analogs and current algebras*, *Lett. Math. Phys.* **15** (1988) 111 [INSPIRE].
- [12] O. Babelon and D. Bernard, *Dressing symmetries*, *Commun. Math. Phys.* **149** (1992) 279 [hep-th/9111036] [INSPIRE].
- [13] A. Ballesteros, E. Celeghini and M.A. del Olmo, *Poisson-Hopf limit of quantum algebras*, *J. Phys. A* **42** (2009) 275202 [arXiv:0903.2178].
- [14] I. Kawaguchi, T. Matsumoto and K. Yoshida, *The classical origin of quantum affine algebra in squashed σ -models*, *JHEP* **04** (2012) 115 [arXiv:1201.3058] [INSPIRE].
- [15] S. Khoroshkin and V. Tolstoy, *Universal R-matrix for quantized (super)algebras*, *Commun. Math. Phys.* **141** (1991) 599.
- [16] E. Twietmeyer, *Real forms of $U_q(\mathfrak{g})$* , *Lett. Math. Phys.* **24** (1992) 49 [INSPIRE].
- [17] V. Chari and A. Pressley, *Quantum groups*, Cambridge University Press, Cambridge U.K. (1994).
- [18] J. Sobczyk, *New real forms of $U_q(\mathfrak{g})$* , in *Spinors, twistors, Clifford algebras and quantum deformations*, Z. Oziewicz et al., Kluwer, The Netherlands (1993).

- [19] H. Ruegg, *Integrable systems, quantum groups, and quantum field theories*, NATO ASI Series Volume 409, Springer, U.S.A. (1993), see p. 45.
- [20] B. Vicedo, *Hamiltonian dynamics and the hidden symmetries of the $AdS_5 \times S^5$ superstring*, *JHEP* **01** (2010) 102 [[arXiv:0910.0221](#)] [[INSPIRE](#)].
- [21] D. Friedan, *Nonlinear models in $2 + \epsilon$ dimensions*, *Phys. Rev. Lett.* **45** (1980) 1057 [[INSPIRE](#)].
- [22] D.H. Friedan, *Nonlinear models in $2 + \epsilon$ dimensions*, *Annals Phys.* **163** (1985) 318 [[INSPIRE](#)].
- [23] F. Delduc, M. Magro and B. Vicedo, *A lattice Poisson algebra for the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring*, *Phys. Lett. B* **713** (2012) 347 [[arXiv:1204.2531](#)] [[INSPIRE](#)].
- [24] R. Asherova, Y.F. Smirnov and V. Tolstoy, *Description of a class of projection operators for semisimple complex Lie algebras*, *Math. Notes Acad. Sci. USSR* **26** (1979) 499.
- [25] V. Tolstoy, *Extremal projectors for Lie algebras and superalgebras of finite growth*, *Russ. Math. Surv.* **44** (1989) 257.