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POSTIVE FIXED POINTS OF LATTICES UNDER SEMIGROUPS OF POSITIVE LINEAR OPERATORS

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Abstract. Let Z be a Banach lattice endowed with positive cone C and an order-continuous norm $\|\cdot\|$. Let G be a semigroup of positive linear endomorphisms of Z. We show that if G is left-reversible (a weaker condition than left-amenability) then the positive fixed points C_0 of Z under G form a lattice cone, and their linear span Z_0 is a Banach lattice under an order-continuous norm $\|\cdot\|_0$ which agrees with $\|\cdot\|$ on C_0. Simple counterexamples show that Z_0 need not contain all the fixed points of Z under G, and need not be a sublattice of (Z, C). Our proof is a simple embedding construction which allows other such results (with different conditions on G) to be read off directly from appropriate fixed point theorems. Results of this kind find application in statistical physics and elsewhere.

Definition 1. A semigroup G is called *left-reversible* iff for all $T_1, T_2 \in G$ there exist $T_3, T_4 \in G$ such that $T_1T_2T_3 = T_2T_1T_4$.

A right ideal of a semigroup G is a set of the form TG where $T \in G$. Left-reversibility of G is equivalent to demanding that every pair of right ideals of G intersect non-trivially. Left-reversibility is a weaker condition than left-amenability for discrete semigroups since the support of any left-invariant mean must be contained in every right ideal. It is strictly weaker since (for example) the free group on two generators is left-reversible (because it is a group) but is not left-amenable (because it is not solvable.) For a survey of the relationships between left-reversibility and other properties of semigroups, see [**6**, §8].

Proposition 2. *Let Z be an order-complete vector lattice with positive cone C, and let G be a semigroup of positive order-continuous linear operators from Z into Z. Let $C_0 = \{x \in C : Tx = x \text{ for all } T \in G\}, Z_0 = C_0 - C_0 = \{x - y : x, y \in C_0\}$.*

If G is left reversible then (Z_0, C_0) is a vector lattice.

Proof. Choose $x, y \in C_0$. Write $x \vee y$ for the least upper bound of x and y in C, and let $A = \{T(x \vee y) : T \in G\}$. Clearly

$$x + y = T(x + y) \geq T(x \vee y) \geq Tx \vee Ty = x \vee y$$

so A is order-bounded above by $x + y$, and hence has a least upper bound z. For $T_1, T_2 \in G$ we have by left-reversibility of G that

$$T_1(x \vee y) \leq T_1T_2T_3(x \vee y) = T_2T_1T_4(x \vee y) \geq T_2(x \vee y)$$

which shows that A is directed as a subset of C, and hence A (considered as a net) is order-convergent to z. The same argument shows that for each $T \in G$, TA is a subnet of A, whence $Tz = z$ and so $z \in C_0$. Clearly $z = x \vee_0 y$, the least upper bound in C_0 of x and y. It follows that C_0 is a lattice cone and hence that Z_0 is a lattice. ∎

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Under the conditions of Proposition 2, Z_0 need not contain all the fixed points of Z under G, and need not be a sublattice of (Z, C), as the following examples show.

Example 3. [**1**, Example 2] Z_0 need not equal the set of all fixed points of Z under G, and this latter set need not be a lattice. Consider R^2 and let G be the semigroup $\{T^n\}$ where $T(x,y) = (2x+y, x+2y)$. The fixed points are the line $x+y=0$ but $C_0 = \{0\}$.

Example 4. [**1**, Example 3] Z_0 need not be a sublattice of (Z, C). Consider R^3 and let G be the semigroup $\{T^n\}$ where $T(x,y,z) = (x,y,x+y)$. The points $a = (1,0,1), b = (0,1,1) \in C_0$ have $a \vee b = (1,1,1), a \vee_0 b = (1,1,2)$.

The conclusion of Proposition 2 may fail if G is not left-reversible, as the following example shows.

Example 5. [**1**, Example 1] Z_0 need not be a lattice for an arbitrary semigroup G. Consider R^5 and define two projections P, Q by

$$P(v,w,x,y,z) = (v,w,x,y,v+w), \quad Q(v,w,x,y,z) = (v,w,x,y,x+y).$$

Let G be the semigroup $\{P, Q\}$, then C_0 is the cone with square base $\{(s, 1-s, t, 1-t, 1) : s,t \in [0,1]\}$ so (Z_0, C_0) is not a lattice. For example $(1,1,2,0,2)$ and $(1,1,1,1,2)$ are minimal upper bounds in Z_0 for $(1,0,1,0,1)$ and $(0,1,1,0,1)$.

Proposition 6. *Let (Z_0, C_0) be a vector lattice. Let Z be a Banach lattice endowed with positive cone C and order-continuous norm $\|\cdot\|$, and suppose that Z_0 can be embedded in Z in such a way that C_0 is a norm closed subset of C.*

Then Z_0 is a Banach lattice with positive cone C_0 and order-continuous norm $\|\cdot\|_0$ defined on Z_0 by $\|x\|_0 = \| \, |x|_0 \|$ where $|\cdot|_0$ is the lattice modulus on (Z_0, C_0).

Proof. Straightforward, for details see the last part of the proof in [**1**, p. 257]. ∎

Again, Z_0 may be a lattice in the order inherited from C but fail to be a sublattice of Z. Conditions under which Z_0 is a sublattice of (Z, C) in Proposition 6 are investigated in [**2**]. Although we always have $\|\cdot\| \le \|\cdot\|_0$ on Z_0, the two norms may differ on non-positive elements of Z_0 such as the element $a-b$ in Example 4, or the more drastic example following.

Example 7. [**1**, Example 4] Z_0 need not be norm closed in $(Z, \|\cdot\|)$, consequently $\|\cdot\|$ and $\|\cdot\|_0$ need not be equivalent on Z_0. Let $Z = l^1$ and define T by

$$(Tw)_{3k} = w_{3k-2} + w_{3k-1} + (1-2^{-k})w_{3k} \quad (Tw)_{3k+1} = w_{3k+1} \quad (Tw)_{3k+2} = w_{3k+2}$$

for $w = (w_k) \in Z$. Let $G = \{T^n\}$ and define $v \in Z$ by

$$v_{3k} = 0; \quad v_{3k+1} = -v_{3k+2} = 1/2^k.$$

Define Z_0 as in Proposition 2. Then v is in the closure of Z_0 under $\|\cdot\|$ but is not in Z_0. This example also satisfies the assertions of Examples 3 and 4.

Propositions 2 and 6 combine to give us

Proposition 8. *Let Z be a Banach lattice endowed with positive cone C and an order-continuous norm $\|\cdot\|$. Let G be a semigroup of positive linear endomorphisms of Z.*

If G is left-reversible then the positive fixed points C_0 of Z under G form a lattice cone, and their linear span Z_0 is a Banach lattice under an order-continuous norm $\|\cdot\|_0$ which agrees with $\|\cdot\|$ on C_0.

Example 5 shows that some condition on G is required. However, we can often use a standard fixed point theorem to recover the conclusion of Proposition 8 for semigroups which are not left-reversible. As an illustration of this, we prove the following:

Definition 9. In the set-up of Proposition 8 call G *norm-distal* iff Gu is norm bounded away from zero for all $u \in Z - \{0\}$.

Proposition 10. *Proposition* 8 *remains true if G is assumed norm-distal in place of left-reversible.*

Proof. Adopting the notation of Proposition 3, pick x, y in C_0 and let A be the smallest subset of C containing x and y and closed under join and orbit, so that for $u, v \in A$ and $T \in G$ we have $u \vee v, Tu \in A$. Now A is directed as a subset of C, and hence convergent to $z = \sup A \leq x + y$. Setting K to be the order interval $[x \vee y, z]$, we have (using order continuity of the norm on Z) that the elements of G act as continuous affine maps from the weakly compact set K into itself [**8**, §2.4]. Since G is distal, K must have a fixed point under G by the Ryall-Nardzewski fixed point theorem [**11**] [**10**]. This fixed point must be z, which is therefore the least upper bound of x and y in C_0. This is true for each choice of x and y in C_0, so C_0 is a lattice cone and the conclusion of Proposition 8 is recovered. ∎

Different variations of Proposition 8 can be obtained by applying other fixed point theorems to the compact convex set K defined in the proof of Proposition 10. See [**5**] for a selection of suitable fixed point properties and [**4**] for a range of recent related work. As well as yielding the new results presented here, our approach also gives simple transparent proofs for a wide range of known results. Properties of this kind find application in statistical mechanics [**9**] [**12**], quantum physics [**3**], statistical decision theory [**7**, Chapter 1] and elsewhere. We conclude this paper with a brief outline of the route to these applications.

Call a convex set S a *simplex* if S can be embedded in a vector lattice Z as a base of the positive cone C. Classical Choquet theory says that if S is a compact metrizable simplex then each point $x \in S$ is the barycentre of exactly one probability measure μ_x supported on $\partial_e S$, the set of extreme points of S. A great deal of work has been devoted to proving similar unique representation results for classes of non-compact or non-metrizable simplices. Lifting the measure norm to Z then makes Z a Banach lattice with $S = \{x \in C : \|x\| = 1\}$.

If the cone C represents the set of states of some process or system (with the convex base S corresponding to the normalized states) then it is frequently desirable to know that S is a simplex (equivalently, that C is a lattice cone) of a type for which a unique representation result is known. For example if the elements of S are the (normalized) Gibbs states of some physical or statistical process, and it is known that S is a simplex of an appropriate kind, then each Gibbs state is uniquely expressible as an average over the set $\partial_e S$ whose elements now correspond to the observables at infinity.

Often we have some simplex S of states or measures, but are interested only in the elements S_0 of S which are fixed under some semigroup G of linear endomorphisms of C, invariance conditions which correspond to physical or observational constraints. The elements of G may be non-conservative or non-stochastic, so need not map S into S even if they have unit spectral radius.

Our results give a simple geometric (rather than measure-theoretic) approach to proving S_0 to be a simplex, for various sets of conditions on G. Provided that the appropriate conditions on S_0 are inherited from S, the fact that S_0 is a simplex then suffices to recover a unique representation theorem for elements of S_0 in terms of extreme points of S_0, i.e. observables of the right kind. Dynkin's entrance boundary can be constructed along similar lines.

The case where Z_0 is not a sublattice of Z perhaps merits more attention than it has received. Here conditional expectations with respect to the tail-field correspond to non-contractive projections.

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