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INVERSE SYSTEMS OF ABSTRACT LEBESGUE SPACES

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ABSTRACT. We show that inverse limits exist in the category of L^1 spaces and positive linear contractions between them. This result generalizes the well-known classical results for inverse systems of Choquet simplexes and of L-balls, but our proof is simple and more purely geometrical. The result finds physical application in the study of random fields.

Let Z be a Banach lattice with positive cone C and norm $\|\cdot\|$. We say that Z is an *AL-space* (short for Abstract Lebesgue space) iff $\|\cdot\|$ is affine on C. See [7, §8, p. 112] for further information about AL-spaces.

Let f be a linear map between two AL-spaces Z_2 and Z_1. We say that f is a *contraction* iff $\|fx\|_1 \leq \|x\|_2$ for all $x \in C_2$. A positive linear contraction $f: Z_2 \to Z_1$ is called an *AL-morphism* .

Note that for an AL-morphism f we have $\|fx\|_1 \leq \|x\|_2$ for all $x \in Z_2$ since

$$\|fx\|_1 = \| |fx|_1 \|_1 \leq \|f|x|_2\|_1 \leq \| |x|_2 \|_2 = \|x\|_2;$$

however, an AL-morphism need not be a lattice homomorphism, an isometry, or even one-to-one.

Let D be a directed set, and suppose that we have an AL-space Z_i associated with each $i \in D$ and an AL-morphism $f_{ji}: Z_i \to Z_j$ associated with each pair $i, j \in D$ where $i > j$.

If this system is consistent in the sense that

$$f_{ji} = f_{jk} f_{ki} \qquad \text{for all } i > k > j \text{ in } D,$$

then we call $(Z_i, f_{ji})_D$ an inverse AL-system.

Now suppose further that Z is an AL-space, and $f_i: Z \to Z_i$ for $i \in D$ are AL-morphisms satisfying

$$f_j = f_{ji} f_i \qquad \text{for all } i > j \text{ in } D.$$

Then we call $(Z, f_i)_D$ a *prefix* of $(Z_i, f_{ji})_D$.

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If every other prefix $(Y, g_i)_D$ of $(Z_i, f_{ji})_D$ factors through $(Z, f_i)_D$ in the sense that there exists an AL-morphism $g: Y \to Z$ with

$$g_i = f_i g \qquad \text{for all } i \in D,$$

then we call $(Z, f_i)_D$ the *inverse AL-limit* of $(Z_i, f_{ji})_D$.

Theorem. *For every inverse AL-system $(Z_i, f_{ji})_D$ the inverse AL-limit $(Z, f_i)_D$ exists and is unique.*

Proof. Clearly the inverse AL-limit is unique up to AL-isomorphism if it exists. We demonstrate existence by constructing the inverse AL-limit explicitly.

Let ΠZ stand for $\prod_{i\in D} Z_i$ and let $\pi_i: \Pi Z \to Z_i$ denote the coordinate projections. Write x_i for $\pi_i x$ where $x \in \Pi Z$.

Define

$$\|x\| = \sup_{i\in D} \|x_i\|_i \quad \text{for } x \in \Pi Z,$$

$$Z = \{x \in \Pi Z: \|x\| < \infty \text{ and } f_{ji} x_i = x_j \text{ for all } i > j \in D\},$$

$$C = \{x \in Z: x_i \in C_i \text{ for all } i \in D\},$$

and let $f_i: Z \to Z_i$ be the restriction of π_i to Z.

We shall show that Z equipped with C and $\|.\|$ is an AL-space. First we note that (Z, C) is a vector lattice with sum and modulus given by

$$(x+y)_j = x_j + y_j, \qquad (|x|)_j = \lim_{i>j} f_{ji}|x_i|_i$$

where $|.|_i$ denotes modulus in Z_i and the limit is in $\|.\|_j$.

To see that the definition of modulus makes sense we use a property of AL-spaces, that every monotone increasing norm-bounded net converges in norm to its least upper bound. This is easy to prove directly, or see [7, 8.2, p. 113].

Clearly $f_{ji}|x_i|_i$ is a monotone increasing sequence in Z_j with norm bounded by $\|x\|$. Hence this sequence is norm convergent in Z_j, and indeed $f_{ji}(|x|)_i = (|x|)_j$ and $\| \, |x| \, \| = \|x\| < \infty$. Thus $|x| \in Z$ and is easily seen to be the least upper bound in Z of $\{x, -x\}$.

It is now easy to verify that $\|.\|$ is a lattice norm for Z and is additive on C. To show the norm complete under these conditions it suffices to prove that every monotone increasing norm bounded net $(x^{(n)})$ in Z has a least upper bound x [7, 8.2, p. 113]. But this is easy, set

$$x_j = \lim_n x_j^{(n)} \quad \text{for } j \in D$$

and argue as in the previous paragraph to show that $x \in Z$ and is the required least upper bound.

This establishes that Z is an AL-space. The f_i are obviously AL-morphisms so $(Z, f_i)_D$ is a prefix of $(Z_i, f_{ji})_D$. For any other prefix $(Y, g_i)_D$ define $g: Y \to Z$ by $(gy)_i = g_i y$. Verification of the factorization property is now routine. Hence $(Z, f_i)_D$ is the inverse AL-limit of $(Z_i, f_{ji})_D$. □

If the dual of a Banach space can be ordered in such a way as to make it an AL-space, then the unit ball of the dual endowed with the weak-$*$ topology is called an L-ball [5].

If all the AL-spaces Z_i are dual spaces of this form then our theorem implies the result of D. A. Edwards [3, Corollary 5, p. 231] that the inverse limit of a system of L-balls is an L-ball.

If in addition each $S_i = \{x \in C_i : \|x\|_i = 1\}$ is a weak-$*$ closed face of the corresponding L-ball, then the S_i are Choquet simplexes.

Conversely (see for example [2, Corollary 3, p. 411]) every Choquet simplex is affine homeomorphic to a simplex of this form. Each affine continuous map f_{ji} from S_i into S_j then extends naturally to an AL-morphism from Z_i into Z_j (although this AL-morphism may fail to be an isometry.)

Thus our theorem also implies the result of F. Jellett, E. B. Davies, and G. F. Vincent-Smith [4, 1] that the inverse limit of a system of Choquet simplexes is a Choquet simplex.

Results of this type are of interest in statistical mechanics (see for example [8, §4.3]). There, the spaces of measures used to underlie the representations of Gibbs states for Föllmer specifications are AL-spaces, and the corresponding substochastic kernals are AL-morphisms.

Often we are concerned only with geometric properties which are intrinsic to the relevent inverse limit space (see for example [6, §1.6]) and in such cases our theorem establishes sufficient structure to determine the phases.

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