Entangled quantum currents in distant mesoscopic Josephson junctions

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Two mesoscopic SQUID rings which are far from each other, are considered. A source of two-mode nonclassical microwaves irradiates the two rings with correlated photons. The Josephson currents are in this case quantum mechanical operators, and their expectation values with respect to the density matrix of the microwaves, yield the experimentally observed currents. Classically correlated (separable) and quantum mechanically correlated (entangled) microwaves are considered, and their effect on the Josephson currents is quantified. Results for two different examples that involve microwaves in number states and coherent states are derived. It is shown that the quantum statistics of the tunnelling electron pairs through the Josephson junctions in the two rings, are correlated.

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I. INTRODUCTION

Superconducting quantum interference devices (SQUIDs) exhibit quantum coherence at the macroscopic level[1]. This is a major research field within condensed matter, and has potential applications in the developing area of quantum information processing[2,3]. A lot of the work on superconducting rings investigates their interaction with classical electromagnetic fields.

In the last twenty years nonclassical electromagnetic fields at low temperatures ($k_{\rm B}T \ll \hbar\omega$) have been studied extensively theoretically and experimentally[4] at both optical and microwave frequencies[5]. They are carefully prepared in a particular quantum state, which is described mathematically with a density matrix ρ. The interaction of SQUID rings with nonclassical microwaves has been studied in the literature[6,7]. In this case the full system, device and microwaves, is quantum mechanical and displays interesting quantum behaviour. For example, the quantum noise in the nonclassical microwaves affects the Josephson currents. Experimental work, which involves the interaction of a Josephson device with a single photon, has recently been reported[8].

An important feature of two-mode nonclassical microwaves is entanglement. Entangled electromagnetic fields have been produced experimentally[9]. There is currently a lot of work on the classification of correlated two-mode electromagnetic fields into classically correlated (separable) and quantum mechanically correlated (entangled)[10]. In a previous publication[11] we have studied the effects of entangled electromagnetic fields on distant electron interference experiments. The interaction of entangled electromagnetic fields with two superconducting charge qubits (that are approximated by two-level systems) has recently been studied[12]. In that work it has been shown that the entanglement is transferred from the photons to the superconducting charge qubits. Related work has also been reported[13]. In this paper we study the effects of entangled electromagnetic fields on the Josephson currents of distant SQUID rings.

We consider two mesoscopic SQUID rings, which are far from each other (Fig. 1). They are irradiated with entangled microwaves, produced by a single source. In this case the phase differences across the Josephson junctions are quantum mechanical operators. Consequently the quantum currents, which are sinusoidal functions of the phase differences, are also operators and their expectation values with respect to the density matrix of the microwaves give the observed Josephson currents. It is shown that for entangled microwaves the currents in the two distant SQUID rings are correlated. We consider suitable examples of separable and entangled microwaves, which differ only by nondiagonal elements; and we show that the correlations between the induced Josephson currents are sensitive to these nondiagonal elements.

In Sec. II we consider a single SQUID ring and study its interaction with nonclassical microwaves. We assume the external field approximation, where the electromagnetic field created by the Josephson current (back reaction) is neglected. We also consider mesoscopic rings which are small in comparison to the wavelength of the microwaves. It is shown that under these assumptions the Josephson current is proportional to the imaginary part of the Weyl function of the nonclassical microwaves.

In Sec. III we analyze the experiment depicted in Fig. 1, where two distant SQUID rings are irradiated with entangled microwaves. We present examples of separable and entangled microwaves that involve number states (Sec. IV) and coherent states (Sec. V). In Sec. VI we present numerical calculations for these examples. In Sec. VII we conclude with a discussion of our results.

II. INTERACTION OF A SINGLE SQUID RING WITH NONCLASSICAL MICROWAVES

In this section we consider a single SQUID ring and study its interaction with both classical and nonclassical microwaves.

A. Classical microwaves

The current is $I_{\rm A} = I_1 \sin\theta_{\rm A}$, where $\theta_{\rm A} = 2e\Phi_{\rm A}$ is the phase difference across the junction due to the total flux $\Phi_{\rm A}$ through the ring. In the external field approximation, $\Phi_{\rm A}$ is simply the externally applied flux, while the back reaction (i.e., the flux induced by the SQUID ring current) is neglected. In other words the flux $\mathcal{L}I_{\rm A}$, where $\mathcal{L}$ is the self-inductance of the ring, is assumed to be much smaller than the external flux $\Phi_{\rm A}$. We consider a magnetic flux with a linear and a sinusoidal component:

$$\Phi_{\rm A} = V_{\rm A}t + \phi_{\rm A}, \quad \phi_{\rm A} = A\sin(\omega_1 t). \tag{1}$$

In this case the current is

$$I_{\rm A} = I_1 \sin[\omega_{\rm A}t + 2eA\sin(\omega_1 t)], \quad \omega_{\rm A} = 2eV_{\rm A}. \tag{2}$$

B. Nonclassical microwaves

In this subsection we consider nonclassical electromagnetic fields, which are carefully prepared in a particular quantum state and are described by a density matrix ρ. In this case, not only the average values $\langle E\rangle, \langle B\rangle$ of the electric and magnetic fields are known, but also the standard deviations $\Delta E, \Delta B$ (and their higher moments).

A particular example is coherent versus squeezed microwaves. In both cases the average values $\langle E\rangle, \langle B\rangle$ are sinusoidal functions of time, that can be made equal for suitable values of the parameters. However the uncertainties are $\Delta E = \Delta B = 2^{-1/2}$ for coherent microwaves; and $\Delta E = \sigma^{-1}2^{-1/2}, \Delta B = \sigma 2^{-1/2}$, for squeezed microwaves (where σ is the squeezing parameter).

Another way of describing nonclassical electromagnetic fields is through the photon counting distribution $P_N = \langle N|\rho|N\rangle$. For example, in the case of coherent and squeezed microwaves the distribution P_N is Poissonian and sub-Poissonian, correspondingly. One of our aims in this paper is to study how the quantum noise in the nonclassical fields, quantified by $\Delta E, \Delta B$, or with the distribution P_N, affects the Josephson currents.

In quantized electromagnetic fields the vector potential A_i and the electric field E_i are dual quantum variables (operators). Strictly speaking the dual quantum variables should be local quantities, but we consider mesoscopic SQUID rings which are much smaller than the wavelength of the microwaves. Therefore we can integrate these quantities over the SQUID ring and obtain the magnetic flux and the electromotive force:

$$\hat{\phi} = \oint_C A_i dx_i, \quad \hat{V}_{\rm EMF} = \oint_C E_i dx_i. \tag{3}$$

As explained above we work in the external field approximation and we neglect the back reaction flux from the electron pairs on the external microwaves. In this case the flux operator evolves as

$$\hat{\phi}(t) = \frac{\xi}{\sqrt{2}}[\hat{a}^\dagger \exp(i\omega t) + \hat{a}\exp(-i\omega t)], \tag{4}$$

where ξ is a parameter proportional to the area of the SQUID ring and the $\hat{a}^\dagger, \hat{a}$ are the photon creation and annihilation operators (e.g., Refs.[4,6]).

In order to go beyond the external field approximation we need to consider the Hamiltonian

$$H = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + H_{\rm SQUID} + H_{\rm int}, \tag{5}$$

where $H_{\rm SQUID}$ is the SQUID Hamiltonian and $H_{\rm int}$ is the interaction term between the SQUID and the microwaves. In this case the flux operator $\hat{\phi}'(t)$ evolves as

$$\begin{aligned} \hat{\phi}'(t) &= \exp(iHt)\hat{\phi}(0)\exp(-iHt) \\ &= \hat{\phi}(t) + \cdots \end{aligned} \tag{6}$$

In this paper we work in the external field approximation and consider the flux operator of Eq. (4).

Consequently the phase difference $\theta_{\rm A}$ is the operator

$$\hat{\theta}_{\rm A} = \omega_{\rm A}t + q[\hat{a}^\dagger\exp(i\omega t) + \hat{a}\exp(-i\omega t)], \; q = \sqrt{2}e\xi; \tag{7}$$

and the current also becomes an operator,

$$\hat{I}_{\rm A} = I_1 \sin\{\omega_{\rm A}t + q[\hat{a}^\dagger\exp(i\omega t) + \hat{a}\exp(-i\omega t)]\}. \tag{8}$$

Expectation values of the current are calculated by taking its trace with respect to the density matrix ρ, which describes the nonclassical electromagnetic fields,

$$\begin{aligned} \langle\hat{I}_{\rm A}\rangle &= {\rm Tr}(\rho\hat{I}_{\rm A}) = I_1{\rm Im}[\exp(i\omega_{\rm A}t)\tilde{W}(\lambda_{\rm A})], \qquad (9)\\ \lambda_{\rm A} &= iq\exp(i\omega_1 t). \qquad (10) \end{aligned}$$

Here $\tilde{W}(x)$ is the Weyl function[14] which is defined in terms of the displacement operator $D(x)$ as

$$\tilde{W}(x) = {\rm Tr}[\rho D(x)]; \quad D(x) = \exp(x\hat{a}^\dagger - x^*\hat{a}). \tag{11}$$

The tilde in the notation of the Weyl function indicates that it is the two-dimensional Fourier transform of the Wigner function.

In a similar way we can calculate the $\langle\hat{I}_{\rm A}^2\rangle = {\rm Tr}(\rho\hat{I}_{\rm A}^2)$. The second (and higher) moments of the current describe the quantum statistics of the electron pairs tunnelling through the Josephson junctions. As explained earlier, nonclassical electromagnetic fields are characterized by the photon counting distribution $P_N = \langle N|\rho|N\rangle$. The statistics of the photons threading the ring affects the statistics of the tunnelling electron pairs, which is quantified with the $\langle\hat{I}_{\rm A}^2\rangle$, $\langle\hat{I}_{\rm A}^3\rangle$, etc.

III. INTERACTION OF TWO DISTANT SQUID RINGS WITH ENTANGLED MICROWAVES

In this section we consider two SQUID rings far apart from each other, which we refer to as A and B (Fig. 1). They are irradiated with microwaves which are produced

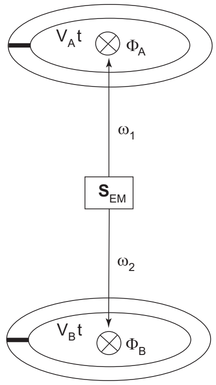

FIG. 1: Two distant mesoscopic SQUID rings A and B are irradiated with nonclassical microwaves of frequencies ω_1 and ω_2, correspondingly. The microwaves are produced by the source $\mathrm{S_{EM}}$ and are correlated. Classical magnetic fluxes $V_{\mathrm{A}}t$ and $V_{\mathrm{B}}t$ are also threading the two rings A and B, correspondingly.

by the same source and are correlated. Let ρ be the density matrix of the microwaves, and

$$\rho_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}\rho, \quad \rho_{\mathrm{B}} = \mathrm{Tr}_{\mathrm{A}}\rho, \tag{12}$$

the density matrices of the microwaves interacting with the two SQUID rings A, B, correspondingly. When the density matrix ρ is factorizable as $\rho_{\mathrm{fact}} = \rho_{\mathrm{A}} \otimes \rho_{\mathrm{B}}$ the two modes are not correlated. If it can be written as $\rho_{\mathrm{sep}} = \sum_i p_i \rho_{\mathrm{A}i} \otimes \rho_{\mathrm{B}i}$, where p_i are probabilities, it is called separable and the two modes are classically correlated. Density matrices which cannot be written in one of these two forms are entangled (quantum mechanically correlated). There has been a lot of work on criteria which distinguish separable and entangled states[10].

The currents in the two SQUIDs are

$$\begin{aligned} \langle \hat{I}_{\mathrm{A}} \rangle &= I_1 \mathrm{Tr}(\rho_{\mathrm{A}} \sin \hat{\theta}_{\mathrm{A}}), && (13) \\ \langle \hat{I}_{\mathrm{B}} \rangle &= I_2 \mathrm{Tr}(\rho_{\mathrm{B}} \sin \hat{\theta}_{\mathrm{B}}). && (14) \end{aligned}$$

The $\langle \hat{I}_{\mathrm{A}} \rangle$ is written in terms of the Weyl function $\tilde{W}(\lambda_{\mathrm{A}})$ in Eq. (9), and similarly for B one may obtain $\langle \hat{I}_{\mathrm{B}} \rangle = I_2 \mathrm{Im}[\exp(i\omega_{\mathrm{B}}t)\tilde{W}(\lambda_{\mathrm{B}})]$, where $\lambda_{\mathrm{B}} = iq\exp(i\omega_2 t)$ and $\omega_{\mathrm{B}} = 2eV_{\mathrm{B}}$.

The expectation value of the product of the two current operators is given by:

$$\langle \hat{I}_{\mathrm{A}} \hat{I}_{\mathrm{B}} \rangle = I_1 I_2 \mathrm{Tr}(\rho \sin \hat{\theta}_{\mathrm{A}} \sin \hat{\theta}_{\mathrm{B}}). \tag{15}$$

We consider the ratio of the currents

$$R = \frac{\langle \hat{I}_{\mathrm{A}} \hat{I}_{\mathrm{B}} \rangle}{\langle \hat{I}_{\mathrm{A}} \rangle \langle \hat{I}_{\mathrm{B}} \rangle}. \tag{16}$$

For factorizable density matrices $\rho_{\mathrm{fact}} = \rho_A \otimes \rho_B$ we easily see that $R_{\mathrm{fact}} = 1$. For separable density matrices $\rho_{\mathrm{sep}} = \sum_i p_i \rho_{\mathrm{A}i} \otimes \rho_{\mathrm{B}i}$ we get

$$R_{\mathrm{sep}} = \frac{\sum_i p_i \langle \hat{I}_{\mathrm{A}i} \rangle \langle \hat{I}_{\mathrm{B}i} \rangle}{(\sum_k p_k \langle \hat{I}_{\mathrm{A}k} \rangle)(\sum_l p_l \langle \hat{I}_{\mathrm{B}l} \rangle)}. \tag{17}$$

We also calculate the second moments

$$\begin{aligned} \langle \hat{I}_{\mathrm{A}}^2 \rangle &= I_1^2 \mathrm{Tr}[\rho_{\mathrm{A}} (\sin \hat{\theta}_{\mathrm{A}})^2], && (18) \\ \langle \hat{I}_{\mathrm{B}}^2 \rangle &= I_2^2 \mathrm{Tr}[\rho_{\mathrm{B}} (\sin \hat{\theta}_{\mathrm{B}})^2]. && (19) \end{aligned}$$

As explained earlier, the statistics of the photons threading the ring affects the statistics of the tunnelling electron pairs, which is quantified with the $\langle \hat{I}_{\mathrm{A}} \hat{I}_{\mathrm{B}} \rangle$, $\langle \hat{I}_{\mathrm{A}}^2 \rangle$, $\langle \hat{I}_{\mathrm{B}}^2 \rangle$, etc.

In the following sections we consider particular examples for the density matrix ρ of the nonclassical microwaves that interact with the two SQUID rings, and examine its effect on these quantities.

IV. MICROWAVES IN NUMBER STATES

We consider microwaves in the separable (mixed) state

$$\rho_{\mathrm{sep}} = \frac{1}{2}(|N_1 N_2\rangle\langle N_1 N_2| + |N_2 N_1\rangle\langle N_2 N_1|), \tag{20}$$

where $N_1 \neq N_2$. We also consider microwaves in the entangled state $|s\rangle = 2^{-1/2}(|N_1 N_2\rangle + |N_2 N_1\rangle)$, which is a pure state. The density matrix of $|s\rangle$ is

$$\rho_{\mathrm{ent}} = \rho_{\mathrm{sep}} + \frac{1}{2}(|N_1 N_2\rangle\langle N_2 N_1| + |N_2 N_1\rangle\langle N_1 N_2|), \tag{21}$$

where the ρ_{sep} is given by Eq. (20). It is seen that the ρ_{ent} and the ρ_{sep} differ only by the above nondiagonal elements, and below we calculate their effect on the Josephson currents.

We note that it is possible to have 'interpolating' density matrices of the form

$$\begin{aligned} \rho_{\mathrm{p}} &= p\rho_{\mathrm{sep}} + (1-p)\rho_{\mathrm{ent}} && (22) \\ &= \rho_{\mathrm{sep}} + \frac{1-p}{2}(|N_1 N_2\rangle\langle N_2 N_1| + |N_2 N_1\rangle\langle N_1 N_2|) \end{aligned}$$

where $0 \leq p \leq 1$. Below we present results for the two extreme cases of ρ_{sep}, where the nondiagonal terms make no contribution; and for the ρ_{ent}, where the nondiagonal terms make maximal contribution. We also present numerical results for the case of ρ_{p}.

In this example, the reduced density matrices are the same for both the separable and entangled states:

$$\begin{aligned} \rho_{\mathrm{sep,A}} &= \rho_{\mathrm{ent,A}} = \rho_{\mathrm{sep,B}} = \rho_{\mathrm{ent,B}} && (23) \\ &= \frac{1}{2}(|N_1\rangle\langle N_1| + |N_2\rangle\langle N_2|). \end{aligned}$$

Consequently in this example $\langle \hat{I}_{\mathrm{A}} \rangle_{\mathrm{sep}} = \langle \hat{I}_{\mathrm{A}} \rangle_{\mathrm{ent}}$, and also $\langle \hat{I}_{\mathrm{B}} \rangle_{\mathrm{sep}} = \langle \hat{I}_{\mathrm{B}} \rangle_{\mathrm{ent}}$.

A. Classically correlated photons

For the density matrix ρ_{sep} of Eq. (20) we find

$$\begin{aligned} \langle \hat{I}_{\mathrm{A}} \rangle &= I_1 C \sin(\omega_{\mathrm{A}} t); && (24) \\ \langle \hat{I}_{\mathrm{B}} \rangle &= I_2 C \sin(\omega_{\mathrm{B}} t); && (25) \\ C &= \frac{1}{2}\exp\left(-\frac{q^2}{2}\right)[L_{N_1}(q^2) + L_{N_2}(q^2)], && (26) \end{aligned}$$

where the $L_n^\alpha(x)$ are Laguerre polynomials (in the case of Eq. (26) we have $\alpha = 0$). The currents $\langle \hat{I}_{\rm A}\rangle, \langle \hat{I}_{\rm B}\rangle$ are in this case independent of the microwave frequencies ω_1, ω_2.

The expectation value of the product of the two currents [Eq. (15)] is

$$\begin{aligned} \langle \hat{I}_{\rm A}\hat{I}_{\rm B}\rangle_{\rm sep} &= I_1 I_2 C_1 \sin(\omega_{\rm A} t)\sin(\omega_{\rm B} t), \quad (27)\\ C_1 &= \exp(-q^2) L_{N_1}(q^2) L_{N_2}(q^2). \quad (28) \end{aligned}$$

Consequently the ratio R of Eq. (16) is

$$R_{\rm sep} = \frac{C_1}{C^2} = \frac{4L_{N_1}(q^2)L_{N_2}(q^2)}{[L_{N_1}(q^2) + L_{N_2}(q^2)]^2}. \quad (29)$$

In this example the $R_{\rm sep}$ is time-independent.

The moments of the currents, defined by Eqs. (18), (19), are also calculated:

$$\begin{aligned} \langle \hat{I}_{\rm A}^2\rangle &= \frac{I_1^2}{2}[1 - C_2\cos(2\omega_{\rm A} t)], \quad (30)\\ \langle \hat{I}_{\rm B}^2\rangle &= \frac{I_2^2}{2}[1 - C_2\cos(2\omega_{\rm B} t)], \quad (31)\\ C_2 &= \frac{1}{2}\exp(-2q^2)[L_{N_1}(4q^2) + L_{N_2}(4q^2)]. \quad (32) \end{aligned}$$

B. Quantum mechanically correlated photons

For the case of $\rho_{\rm ent}$ the reduced density matrices $\rho_{\rm A}, \rho_{\rm B}$ are those given by Eq. (23), and consequently the $\langle \hat{I}_{\rm A}\rangle, \langle \hat{I}_{\rm B}\rangle$ are the same as in Eqs. (24), (25); and the $\langle \hat{I}_{\rm A}^2\rangle, \langle \hat{I}_{\rm B}^2\rangle$ are the same as in Eqs. (30), (31).

However in this case the $\langle \hat{I}_{\rm A}\hat{I}_{\rm B}\rangle$ is

$$\langle \hat{I}_{\rm A}\hat{I}_{\rm B}\rangle_{\rm ent} = \langle \hat{I}_{\rm A}\hat{I}_{\rm B}\rangle_{\rm sep} + I_{\rm cross}, \quad (33)$$

where

$$\begin{aligned} I_{\rm cross} = &-I_1 I_2 C_3 \cos[(N_1 - N_2)(\omega_1 - \omega_2)t]\\ &\times[\cos(\omega_{\rm A} t + \omega_{\rm B} t) - (-1)^{N_1-N_2}\cos(\omega_{\rm A} t - \omega_{\rm B} t)], \end{aligned} \quad (34)$$

$$C_3 = \frac{1}{2}\exp(-q^2) L_{N_1}^{N_2-N_1}(q^2) L_{N_2}^{N_1-N_2}(q^2). \quad (35)$$

The term $I_{\rm cross}$ is induced by the nondiagonal elements of $\rho_{\rm ent}$, and depends on the photon frequencies ω_1, ω_2. This term quantifies the difference between the effect of separable and entangled microwaves on the Josephson currents. We note that the nondiagonal terms of $\rho_{\rm ent}$ [Eq. (21)] are small and consequently they are very sensitive to the back reaction. Therefore our results which neglect the back reaction are relevant to experiments with small Josephson currents. In other words it is required that the fluxes $\mathcal{L}_{\rm A} I_{\rm A}$ and $\mathcal{L}_{\rm B} I_{\rm B}$ are much smaller than the external flux.

The ratio R of Eq. (16) can be simplified in two distinct expressions according to whether the difference $N_1 - N_2$ is even or odd. In the case $N_1 - N_2 = 2k$, the ratio is

$$R_{\rm ent}^{(2k)} = R_{\rm sep} + \frac{4L_{N_1}^{-2k}(q^2)L_{N_2}^{2k}(q^2)}{[L_{N_1}(q^2) + L_{N_2}(q^2)]^2}\cos(\Omega t), \quad (36)$$

where

$$\Omega = (N_1 - N_2)(\omega_1 - \omega_2). \quad (37)$$

It is seen that the $R_{\rm ent}^{(2k)}$ oscillates around the $R_{\rm sep}$ with frequency Ω given by Eq. (37). If there is no detuning between the nonclassical electromagnetic fields, i.e. $\omega_1 = \omega_2$, then $R_{\rm ent}^{(2k)}$ is constant, although it is still $R_{\rm ent} \neq R_{\rm sep}$.

In the case $N_1 - N_2 = 2k + 1$ the ratio is

$$\begin{aligned} R_{\rm ent}^{(2k+1)} &= R_{\rm sep}\\ &- \frac{4L_{N_1}^{-2k-1}(q^2)L_{N_2}^{2k+1}(q^2)}{[L_{N_1}(q^2) + L_{N_2}(q^2)]^2}\frac{\cos(\Omega t)}{\tan(\omega_{\rm A} t)\tan(\omega_{\rm B} t)}. \end{aligned} \quad (38)$$

In both cases the $R_{\rm ent}$ is time-dependent and it is a function of the photon frequencies ω_1, ω_2, in contrast to the case of $R_{\rm sep}$ (which is time-independent).

V. MICROWAVES IN COHERENT STATES

We consider microwaves in the classically correlated state

$$\rho_{\rm sep} = \frac{1}{2}(|A_1 A_2\rangle\langle A_1 A_2| + |A_2 A_1\rangle\langle A_2 A_1|). \quad (39)$$

The $|A_1\rangle$, $|A_2\rangle$ are microwave coherent states (eigenstates of the annihilation operators). We also consider the entangled state $|u\rangle = \mathcal{N}(|A_1 A_2\rangle + |A_2 A_1\rangle)$, with density matrix

$$\begin{aligned} \rho_{\rm ent} &= 2\mathcal{N}^2\rho_{\rm sep}\\ &+ \mathcal{N}^2(|A_1 A_2\rangle\langle A_2 A_1| + |A_2 A_1\rangle\langle A_1 A_2|), \quad (40) \end{aligned}$$

where the normalization constant is given by

$$\mathcal{N} = \left[2 + 2\exp\left(-|A_1 - A_2|^2\right)\right]^{-1/2}. \quad (41)$$

A. Classically correlated photons

For microwaves in the separable state of Eq. (39) the reduced density matrices are

$$\rho_{\rm sep,A} = \rho_{\rm sep,B} = \frac{1}{2}(|A_1\rangle\langle A_1| + |A_2\rangle\langle A_2|), \quad (42)$$

and hence the currents in A and B are

$$\begin{aligned} \langle \hat{I}_{\rm A}\rangle_{\rm sep} &= \frac{I_1}{2}\exp(-\frac{q^2}{2})\{\sin[\omega_{\rm A} t + 2q|A_1|\cos(\omega_1 t - \theta_1)]\\ &+ \sin[\omega_{\rm A} t + 2q|A_2|\cos(\omega_1 t - \theta_2)]\}, \quad (43) \end{aligned}$$

$$\langle \hat{I}_{\rm B}\rangle_{\rm sep} = \frac{I_2}{2}\exp(-\frac{q^2}{2})\{\sin[\omega_{\rm B}t + 2q|A_1|\cos(\omega_2 t-\theta_1)] + \sin[\omega_{\rm B}t + 2q|A_2|\cos(\omega_2 t-\theta_2)]\}, \quad (44)$$

where $\theta_1 = \arg(A_1)$, and $\theta_2 = \arg(A_2)$. We have also calculated numerically the ratio $R_{\rm sep}$.

B. Quantum mechanically correlated photons

For microwaves in the entangled state of Eq. (40) the reduced density matrices are

$$\rho_{\rm ent,A} = \rho_{\rm ent,B} = \mathcal{N}^2(|A_1\rangle\langle A_1| + |A_2\rangle\langle A_2| + \tau|A_1\rangle\langle A_2| + \tau^*|A_2\rangle\langle A_1|), \quad (45)$$

where

$$\tau = \langle A_1|A_2\rangle = \exp\left(-\frac{|A_1|^2}{2} - \frac{|A_2|^2}{2} + A_1^* A_2\right). \quad (46)$$

The current in A is

$$\langle \hat{I}_{\rm A}\rangle_{\rm ent} = 2\mathcal{N}^2\langle \hat{I}_{\rm A}\rangle_{\rm sep} + \mathcal{N}^2 E F_1 \exp\left(-\frac{q^2}{2}\right) I_1, \quad (47)$$

where

$$E = \exp[-|A_1|^2 - |A_2|^2 + 2|A_1 A_2|\cos(\theta_1-\theta_2)], \quad (48)$$

$$F_1 = \{\exp[q|A_1|S_{A,1}(t) - q|A_2|S_{A,2}(t)] + \exp[-q|A_1|S_{A,1}(t) + q|A_2|S_{A,2}(t)]\} \times \sin[\omega_A t + q|A_1|C_{A,1}(t) + q|A_2|C_{A,2}(t)], \quad (49)$$

and

$$S_{A,1} = \sin(\omega_1 t-\theta_1), \quad S_{A,2} = \sin(\omega_1 t - \theta_2),$$
$$C_{A,1} = \cos(\omega_1 t-\theta_1), \quad C_{A,2} = \cos(\omega_1 t - \theta_2). \quad (50)$$

The current in B is

$$\langle \hat{I}_{\rm B}\rangle_{\rm ent} = 2\mathcal{N}^2\langle \hat{I}_{\rm B}\rangle_{\rm sep} + \mathcal{N}^2 E F_2 \exp\left(-\frac{q^2}{2}\right) I_2, \quad (51)$$

where

$$F_2 = \{\exp[q|A_1|S_{B,1}(t) - q|A_2|S_{B,2}(t)] + \exp[-q|A_1|S_{B,1}(t) + q|A_2|S_{B,2}(t)]\} \times \sin[\omega_B t + q|A_1|C_{B,1}(t) + q|A_2|C_{B,2}(t)], \quad (52)$$

and

$$S_{B,1} = \sin(\omega_2 t-\theta_1), \quad S_{B,2} = \sin(\omega_2 t - \theta_2),$$
$$C_{B,1} = \cos(\omega_2 t-\theta_1), \quad C_{B,2} = \cos(\omega_2 t - \theta_2). \quad (53)$$

We have also calculated numerically the ratio $R_{\rm ent}$.

As we already explained, the nondiagonal terms in $\rho_{\rm ent}$ are very sensitive to back reaction and therefore these results are relevant to experiments with small Josephson currents. In other words it is required that the fluxes $\mathcal{L}_{\rm A} I_{\rm A}$ and $\mathcal{L}_{\rm B} I_{\rm B}$ are much smaller than the external flux.

VI. NUMERICAL RESULTS

In all numerical results of Figs. 2 to 6 the microwave frequencies are $\omega_1 = 1.2\times10^{-4}, \omega_2 = 10^{-4}$, in units where $k_B = \hbar = c = 1$. The critical currents are $I_1 = I_2 = 1$. The other parameters are $\xi = 1$, $\omega_{\rm A} = \omega_1$, $\omega_{\rm B} = \omega_2$, $N_1 = 2$, $N_2 = 0$, and the arguments of the coherent eigenstates are $\theta_1 = \theta_2 = 0$. For a meaningful comparison between microwaves in number states and microwaves in coherent states, we take $|A_1|^2 = N_1$ and $|A_2|^2 = N_2$, so that the average number of photons in the coherent states is equal to the number of photons in the number states.

In Fig. 2 we plot $R_{\rm sep}$ against $(\omega_1 - \omega_2)t$ for currents induced by microwaves in the number state of Eq. (20) with $N_1 = 2, N_2 = 0$ (line of circles), and the coherent state of Eq. (39) with $A_1 = \sqrt{2}, A_2 = 0$ (solid line). It is seen that two different microwave states with the same average number of photons give different results on the quantum statistics of the electron pairs.

In Fig. 3 we plot $R_{\rm sep} - R_{\rm ent}$ against $(\omega_1 - \omega_2)t$ for currents induced by microwaves in (a) the number states of Eqs. (20) and (21) with $N_1 = 2, N_2 = 0$, and (b) the coherent states of Eqs. (39) and (40) with $A_1 = \sqrt{2}, A_2 = 0$. It is seen that the separable and entangled states, which differ only by nondiagonal elements, give different results. As expected, the difference (which shows the effect of the nondiagonal elements) is small, but it is nonzero.

In Fig. 4 we plot (a) $\langle \hat{I}_{\rm A}\rangle_{\rm sep} - \langle \hat{I}_{\rm A}\rangle_{\rm ent}$, and (b) $\langle \hat{I}_{\rm A}^2\rangle_{\rm sep} - \langle \hat{I}_{\rm A}^2\rangle_{\rm ent}$, against $(\omega_1 - \omega_2)t$ for microwaves in the coherent state $\rho_{\rm A,sep}$ of Eq. (42) and $\rho_{\rm A,ent}$ of Eq. (45) with $A_1 = \sqrt{2}, A_2 = 0$. For coherent states $\rho_{\rm ent,A}$ is not equal to $\rho_{\rm sep,A}$ [cf. Eqs. (45), (42)] and consequently the corresponding currents are different. For number states, the currents corresponding to $\rho_{\rm ent,A}$ and $\rho_{\rm sep,A}$ are the same because $\rho_{\rm ent,A} = \rho_{\rm sep,A}$ [cf. Eq. (23)].

In Fig. 5 we plot $\langle \hat{I}_{\rm A}\hat{I}_{\rm B}\rangle_{\rm sep} - \langle \hat{I}_{\rm A}\hat{I}_{\rm B}\rangle_{\rm ent}$ for (a) number states of Eqs. (20) and (21) with $N_1 = 2, N_2 = 0$, and (b) coherent states of Eqs. (39) and (40) with $A_1 = \sqrt{2}, A_2 = 0$, against $(\omega_1 - \omega_2)t$. In this figure also, we get different results due to the nondiagonal elements in the entangled state.

In Fig. 6 we plot the ratio $R_{\rm p}$ of Eq. (16) against $p \in [0, 1]$ for currents that are induced by the interpolating density matrix $\rho_{\rm p}$ of Eq. (22) for number states with $N_1 = 2, N_2 = 0$. The time t has been fixed so that $(\omega_1 - \omega_2)t = \pi/2$.

VII. DISCUSSION

We have considered the interaction of SQUID rings with nonclassical microwaves. We have assumed the external field approximation, where the electromagnetic field created by the Josephson currents (back reaction) is neglected. We have also considered small rings in comparison to the wavelength of the microwaves and taken as dual quantum variables the magnetic flux and electromotive force of Eq. (3).

The Josephson current is an operator and its expectation value with respect to the density matrix of the nonclassical microwaves is the observed current. We have shown that the expectation value of the current is proportional to the imaginary part of the Weyl function [Eq. (9)]. This shows clearly how the full density matrix of the microwaves affects the Josephson current. The higher moments of the current $\langle \hat{I}_{\rm A}^M \rangle$ can also be calculated and used to quantify the statistics of the tunnelling electron pairs. It has been shown that the statistics of the irradiating photons determine the tunnelling statistics of the electron pairs.

We have also considered the interaction of two distant SQUID rings A and B with two-mode nonclassical microwaves, which are produced by the same source. It has been shown that classically correlated (separable) and quantum mechanically correlated (entangled) photons induce different Josephson currents and different tunnelling statistics in the two devices. The results show that the entangled photons produce entangled Josephson currents in the distant SQUID rings. This can have applications in the general area of quantum information processing.

The work can be extended in various directions. The first is to take into account the back reaction and include the extra terms which we have neglected in Eq. (6). This can be done numerically. The second direction is the study of Bell-like inequalities for the Josephson currents; which are violated when the currents are entangled. Another direction is the potential use of the system as a detector of entangled photons. There is a lot of work on the use of mesoscopic devices as detectors[15]. Application of the present work in this direction requires further work, which could lead to the development of a detection system for entangled photons based on two distant SQUID rings.

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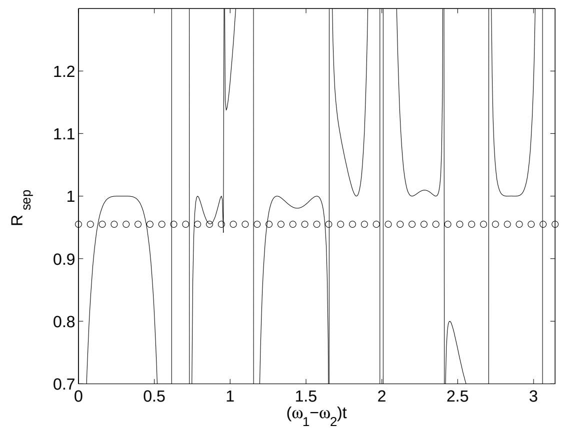

FIG. 2: R_{sep} against $(\omega_1 - \omega_2)t$ for the number state of Eq. (20) with $N_1 = 2, N_2 = 0$ (line of circles), and the coherent state of Eq. (39) with $A_1 = \sqrt{2}, A_2 = 0$ (solid line). The photon frequencies are $\omega_1 = 1.2 \times 10^{-4}$ and $\omega_2 = 10^{-4}$, in units where $k_B = \hbar = c = 1$.

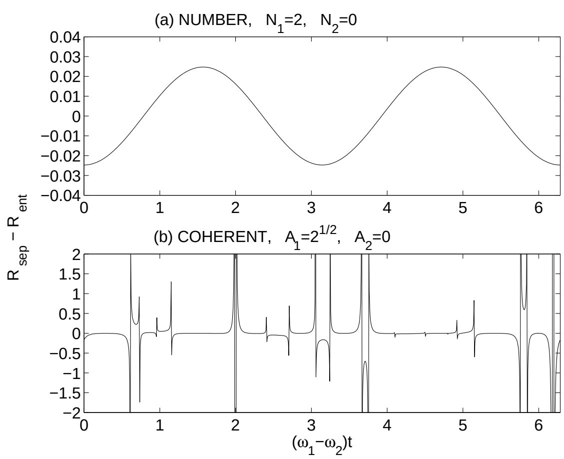

FIG. 3: $R_{\text{sep}} - R_{\text{ent}}$ against $(\omega_1 - \omega_2)t$ for (a) number states of Eqs. (20) and (21) with $N_1 = 2, N_2 = 0$, and (b) coherent states of Eqs. (39) and (40) with $A_1 = \sqrt{2}, A_2 = 0$. The photon frequencies are $\omega_1 = 1.2 \times 10^{-4}$ and $\omega_2 = 10^{-4}$, in units where $k_B = \hbar = c = 1$.

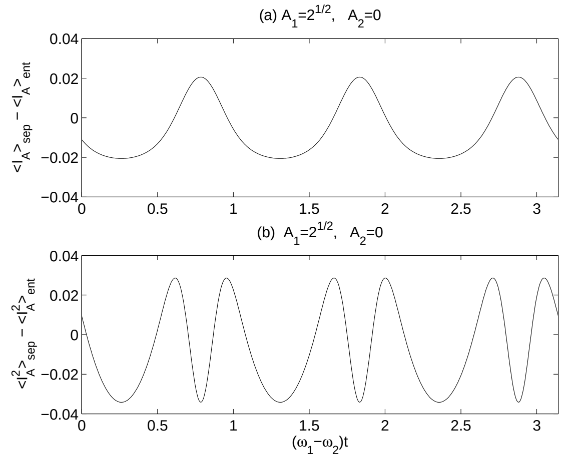

FIG. 4: (a) $\langle \hat{I}_\mathrm{A} \rangle_\mathrm{sep} - \langle \hat{I}_\mathrm{A} \rangle_\mathrm{ent}$, and (b) $\langle \hat{I}_\mathrm{A}^2 \rangle_\mathrm{sep} - \langle \hat{I}_\mathrm{A}^2 \rangle_\mathrm{ent}$ against $(\omega_1 - \omega_2)t$ for the coherent state $\rho_{\mathrm{sep,A}}$ of Eq. (42) and $\rho_{\mathrm{ent,A}}$ of Eq. (45) with $A_1 = \sqrt{2}, A_2 = 0$. The photon frequencies are $\omega_1 = 1.2 \times 10^{-4}$ and $\omega_2 = 10^{-4}$, in units where $k_B = \hbar = c = 1$.

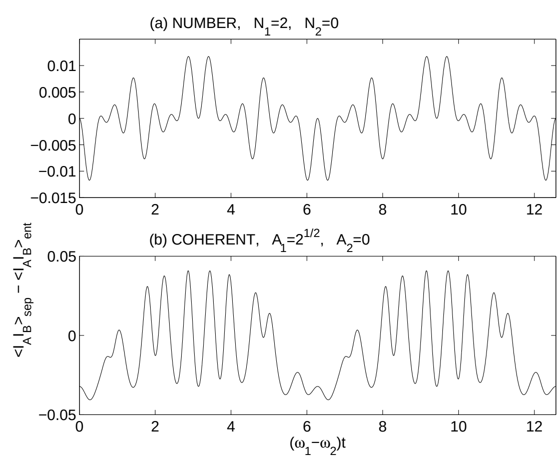

FIG. 5: $\langle \hat{I}_\mathrm{A} \hat{I}_\mathrm{B} \rangle_\mathrm{sep} - \langle \hat{I}_\mathrm{A} \hat{I}_\mathrm{B} \rangle_\mathrm{ent}$ for (a) number states of Eqs. (20) and (21) with $N_1 = 2, N_2 = 0$, and (b) coherent states of Eqs. (39) and (40) with $A_1 = \sqrt{2}, A_2 = 0$, against $(\omega_1 - \omega_2)t$. The photon frequencies are $\omega_1 = 1.2 \times 10^{-4}$ and $\omega_2 = 10^{-4}$, in units where $k_B = \hbar = c = 1$.

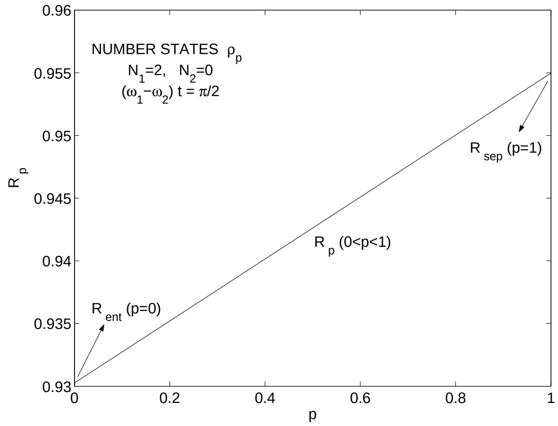

FIG. 6: $R_{\rm p}$ against p for currents that are induced by the interpolating density matrix $\rho_{\rm p}$ of Eq. (22) for number states with $N_1 = 2, N_2 = 0$. The time t has been fixed so that $(\omega_1 - \omega_2)t = \pi/2$. The photon frequencies are $\omega_1 = 1.2 \times 10^{-4}$ and $\omega_2 = 10^{-4}$, in units where $k_B = \hbar = c = 1$.